General Relativity
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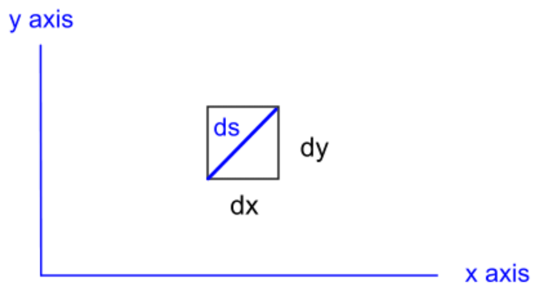
Chapter GR-2. The Metric Tensor

GR2-1. The Differential Line Element in Cartesian and Polar Coordinates. You may have heard that general relativity deals with curved spacetime. In this chapter we will study flat and curved two-dimensional surfaces as we categorize the information needed to calculate line elements. The result will be a nice compact form known as the metric tensor. I know you are anxious to get to tensors, but our slow approach below will give you much insight.

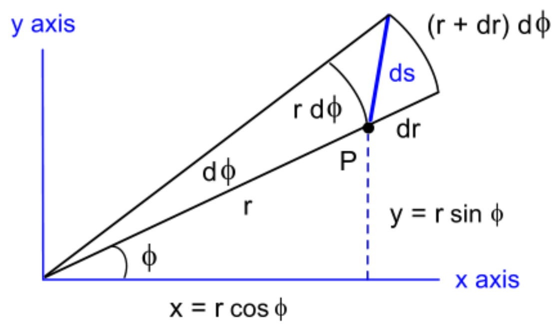
The distance between two points, where the sides are orthogonal, is given by the Pythagorean formula.



In Cartesian coordinates, the square of the differential line element is

$$ds^2 = dx^2 + dy^2 .$$

If we use polar coordinates instead, we can write

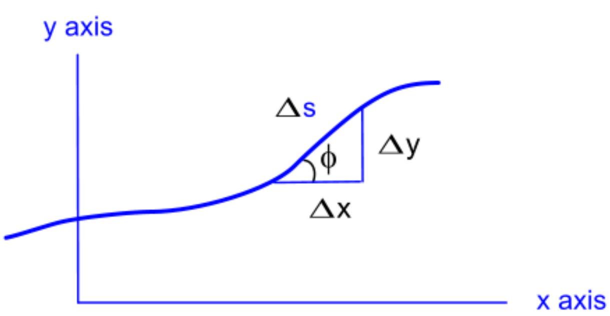


$$ds^2 = dr^2 + r^2 d\phi^2 ,$$

as we know from the previous chapter or referring to the diagram at the left.

Both of these surfaces are flat. Eventually, we will want a test that we can do to determine whether a space is flat or not.

GR2-2. Arc Length. Before considering curved two dimensional surfaces, let's look at curvature in one dimension. This section will be a review of things you covered in differential calculus.



First use deltas instead of differentials to play it safe. We can write

$$\Delta s^2 = \Delta x^2 + \Delta y^2 \text{ and}$$

$$\Delta s^2 = (1 + \frac{\Delta y^2}{\Delta x^2})\Delta x^2 .$$

Then, eventually taking the limit as $\Delta x \to 0$, we have

$$\Delta s^2 = (1 + \frac{\Delta y^2}{\Delta x^2})\Delta x^2 = (1 + \left[\frac{\Delta y}{\Delta x}\right]^2)\Delta x^2 \to (1 + \left[\frac{dy}{dx}\right]^2)dx^2$$, i.e.,

$$ds^2 = \lim_{\Delta x \to 0} \Delta s^2 = \lim_{\Delta x \to 0} (1 + \left[\frac{\Delta y}{\Delta x}\right]^2)\Delta x^2 = (1 + \left[\frac{dy}{dx}\right]^2)dx^2$$

With the usual shorthand notation for the derivative, $y' = \frac{dy}{dx}$, we obtain

$$ds^2 = (1 + y'^2)dx^2 \quad \text{and} \quad ds = \sqrt{(1 + y'^2)}dx .$$

The arc length of a function y(x) from point 1 to 2 is provided by

$$s = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{(1 + y'^2)}dx$$

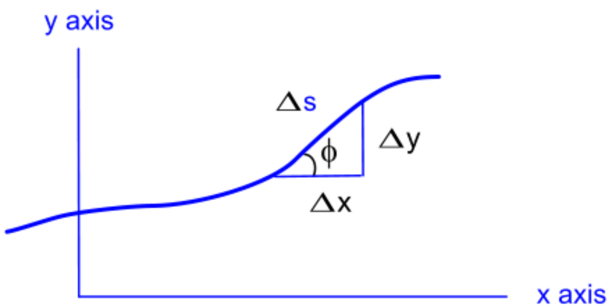
We can call this the formula for arc length.

Homework HW-10. The Circumference of a Circle. The equation for a circle of radius R and center $(0,0)$ is given by

$$R^2 = x^2 + y^2 .$$

Use the arc length formula to show that the arc length in the first quadrant is $\frac{\pi}{2}R$, which means the circumference is $4\frac{\pi}{2}R = 2\pi R$. You might consider the trig substitution $x = R\cos\theta$ when doing your integral.

GR2-3. The Curvature of a Line. Curvature is a measure of the change in direction as one proceeds along a given curve.



Think of slope as $\frac{rise}{run} = \frac{\Delta y}{\Delta x}$ in the limit as Δx approaches zero. So the slope is given by the first derivative:

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \tan\phi \,.$$

When there is a change in slope, there is curvature. Therefore, we expect the curvature to depend on the second derivative, which gives us a measure of the change in slope.

$$y'' = \lim_{\Delta x \to 0} \frac{\Delta y'}{\Delta x} = \frac{d^2 y}{dx^2} \Rightarrow \frac{change\ in\ slope}{run} \,.$$

However, the curvature is not defined simply as y'' and there is good reason for this, which will be explained in this chapter. Instead, the curvature is defined in terms of the change in angle ϕ relative a step along the arc length.

$$K = \left| \lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s} \right| = \left| \frac{d\phi}{ds} \right|$$

We take the absolute value so that a curve bending up or down by the same degree has the same curvature.

For calculational purposes, it is easier to take derivatives of y with respect to x rather than derivatives of ϕ with respect to s. So we use the chain rule to lead us to an equivalent expression for K.

$$K = \left| \frac{d\phi}{ds} \right| = \left| \frac{d\phi}{dx} \frac{dx}{ds} \right|$$

Note from the above figure that $\tan\phi = \frac{dy}{dx}$. We have the following three formulas to work with.

$$ds = \sqrt{(1+y'^2)}dx \qquad \tan\phi = \frac{dy}{dx} \qquad K = \left|\frac{d\phi}{ds}\right| = \left|\frac{d\phi}{dx}\frac{dx}{ds}\right|$$

First note that

$$\frac{ds}{dx} = \sqrt{(1+y'^2)} \quad \text{and the flip} \quad \frac{dx}{ds} = \frac{1}{\sqrt{(1+y'^2)}} .$$

The last formula can be substituted in the K formula:

$$K = \left|\frac{d\phi}{ds}\right| = \left|\frac{d\phi}{dx}\frac{dx}{ds}\right| = \left|\frac{d\phi}{dx}\frac{1}{\sqrt{(1+y'^2)}}\right| .$$

We can find $\frac{d\phi}{dx}$ from $\phi = \tan^{-1}\frac{dy}{dx}$. Let $u = \frac{dy}{dx}$ so $\frac{d\tan^{-1}u}{dx} = \frac{1}{1+u^2}$. Then,

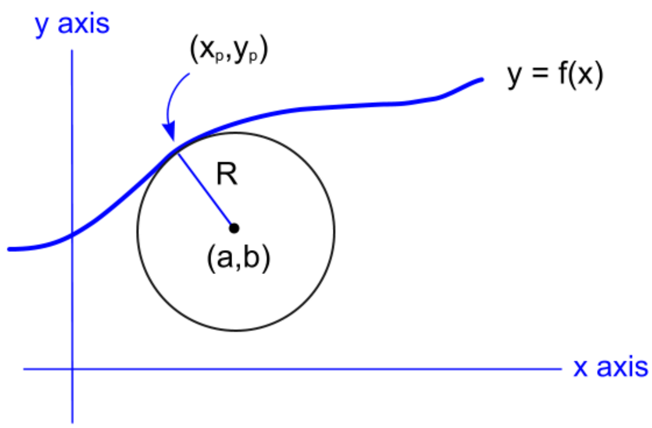
$$\frac{d\phi}{dx} = \frac{1}{1+y'^2}\frac{dy'}{dx} = \frac{1}{1+y'^2}y'' .$$

Note the appearance of the second derivative as we expected. Putting all this together, we find

$$K = \left|\frac{d\phi}{dx}\frac{1}{\sqrt{(1+y'^2)}}\right| = \left|\frac{1}{1+y'^2}y''\frac{1}{\sqrt{(1+y'^2)}}\right|$$, leading to the compact result

$$K = \frac{|y''|}{(1+y'^2)^{3/2}} .$$

Our definition for the curvature, $K = \frac{|y''|}{(1+y'^2)^{3/2}}$, has interesting geometric meaning. Consider the curve $y = f(x)$ with a particular point (x_p, y_p) singled out and a circle centered at (a,b) that touches the curve tangentially at the designated point (x_p, y_p).



Two important properties of the circle are:

a) The circle touches the curve at point (x_p, y_p). Therefore,

$$(x_p - a)^2 + (y_p - b)^2 = R^2 .$$

b) The circle is tangent at point (x_p, y_p). Thus, the slope of the curve $y' = f'(x)$ matches the slope of the circle at that point.

$$y'_p = f'(x_p) = -\left[\frac{x_p - a}{y_p - b}\right].$$

Do you recognize the trick from high-school algebra?

Here is what we did. The slope from the center of the circle up the radius line shown is $m = -\left[\frac{y_p - b}{a - x_p}\right]$, with a negative sign since the line is a "sliding board."
The slope of the curve at p is perpendicular. The old high-school trick is that the product of perpendicular slopes equals -1. Therefore, for the slope of the curve at point p, which we will call m_p, must satisfy $m_p m = -1$. This equation leads us to

$$m_p = -\left[\frac{x_p - a}{y_p - b}\right].$$

Homework HW-11. The Slope of a Circle. Show by implicit differentiation of $(x-a)^2+(y-b)^2=R^2$ that you arrive at $\frac{dy}{dx}=-\left[\frac{x-a}{y-b}\right]$ for the circle, leading to our same result above: $m_p=\left.\frac{dy}{dx}\right|_p=-\left[\frac{x_p-a}{y_p-b}\right]$.

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But many circles with different radii satisfy our two criteria:

$$(x_p-a)^2+(y_p-b)^2=R^2,$$

$$y'_p=f'(x_p)=-\left[\frac{x_p-a}{y_p-b}\right].$$

In order to find the circle that fits the best at point (x_p, y_p) we demand that the changes in slopes of $y=f(x)$ and the circle also agree at our point. This gives us the third restriction for the circle. The second derivative for the circle is found taking the derivative of your HW-11 homework assignment result $\frac{dy}{dx}=-\left[\frac{x-a}{y-b}\right]$. Proceeding to take this derivative, we obtain

$$\frac{d^2y}{dx^2}=-\frac{d}{dx}\left[\frac{x-a}{y-b}\right]=-\frac{1}{y-b}+\frac{(x-a)}{(y-b)^2}y'.$$

Using $y'=-\left[\frac{x-a}{y-b}\right]$, we note $\frac{d^2y}{dx^2}=-\frac{1}{y-b}-\frac{y'}{(y-b)}y'$, which leads to

$$y''=-\left[\frac{1+y'^2}{y-b}\right].$$

At our designated point this second derivative must match $y''=f''(x)$.

We have three unknowns for our circle: a, b, and R. But we have three equations. These equations are listed below where it is understood that everything is to be applied at point (x_p, y_p).

$$(x-a)^2 + (y-b)^2 = R^2$$

$$y' = f'(x) = -\left[\frac{x-a}{y-b}\right]$$

$$y'' = -\left[\frac{1+y'^2}{y-b}\right]$$

Homework HW-12. The Radius of Curvature. Solve the above equations for the radius R and show that

$R = \dfrac{(1+y'^2)}{|y''|}$, which is the inverse of our curvature $K = \dfrac{|y''|}{(1+y'^2)^{3/2}}$.

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Your homework assignment shows us that the curvature matches that of our circle.

$$K = \frac{|y''|}{(1+y'^2)^{3/2}} = \frac{1}{R}$$

Now you can see why the curvature is defined this way rather than simply y''.

For a straight line $y' = const$ and $y'' = 0$. These results lead to zero curvature and a radius of curvature that is infinite.

Homework HW-13. The Radius of Curvature for an Ellipse. The equation of an ellipse with semi-major axis a and semi-minor axis b is $\left[\dfrac{x}{a}\right]^2 + \left[\dfrac{y}{b}\right]^2 = 1$. Show that the curvature $K = \frac{ab}{(a^2 - \varepsilon^2 x^2)^{3/2}}$, where $\varepsilon^2 = 1 - \frac{b^2}{a^2}$. The parameter ε is known as the eccentricity. Note that for a circle, $\varepsilon = 0$ as $a = b$ and

$$K = \frac{ab}{(a^2 - \varepsilon^2 x^2)^{3/2}} \to \frac{R^2}{(R^2)^{3/2}} = \frac{R^2}{R^3} = \frac{1}{R}.$$

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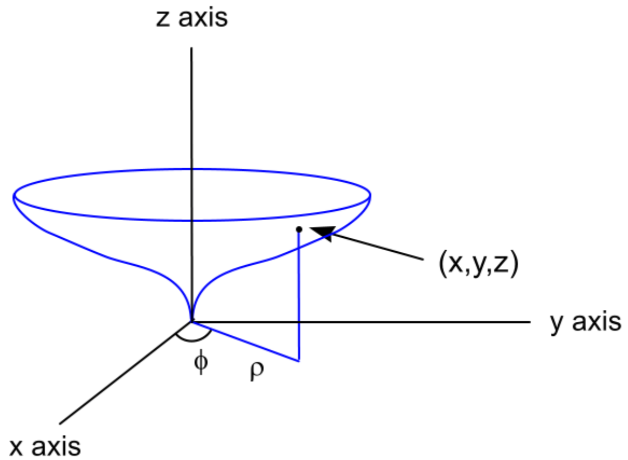
GR2-4. The Curvature of a Surface. Due to the symmetry found in problems of interest to us, we will consider surfaces of revolution.



We choose the z-axis as the axis of revolution. The Cartesian coordinates (x,y,z) expressed in terms of cylindrical coordinates (ρ,ϕ,z) are

$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$

$$z = f(\rho).$$

We can call the function $f(\rho)$ the profile curve for the surface. This profile is clearly visible in the z-y plane where $\phi = 90^\circ$ and $\rho = y$.

The square of the differential line element in cylindrical coordinates is

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2.$$

On the surface $z = f(\rho)$ and $dz = \frac{df(\rho)}{d\rho} d\rho = f' d\rho$, which leads to

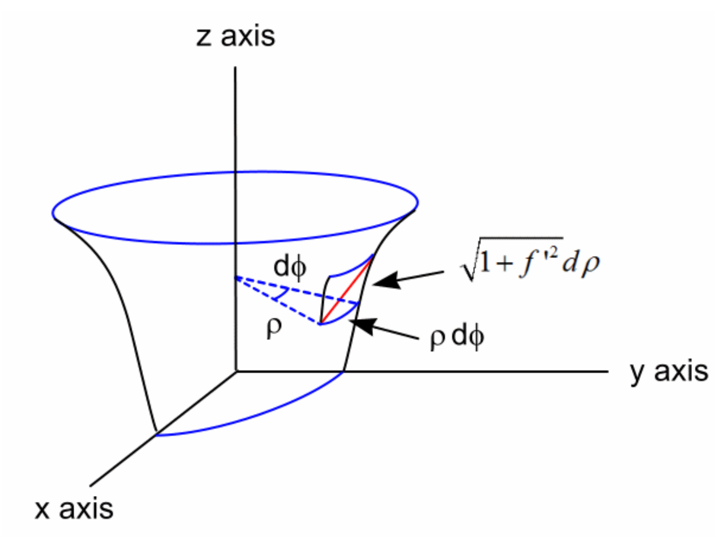
$$ds^2 = (1 + f'^2) d\rho^2 + \rho^2 d\phi^2.$$

This square of the line element on the surface is called the first fundamental form of the surface.

A visualization of the first fundamental of the surface,

$$ds^2 = (1 + f'^2)d\rho^2 + \rho^2 d\phi^2$$

is illustrated by the figure below. Note that ds is the red segment in the figure.



You can notice the similarity when we wrote earlier for a curve in a plane

$$ds = \sqrt{(1 + y'^2)}\,dx \, .$$

We see a piece like that in the form

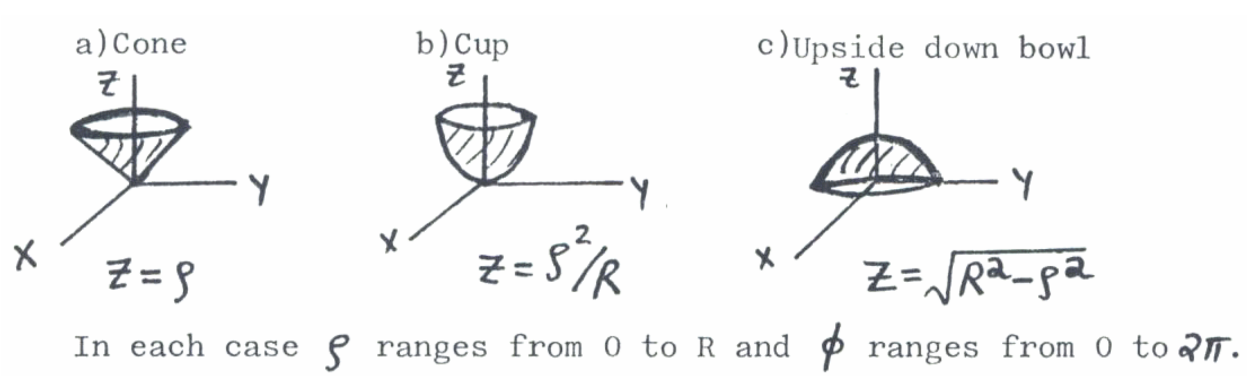
$$\sqrt{(1 + f'^2)}\,d\rho$$

along the surface.

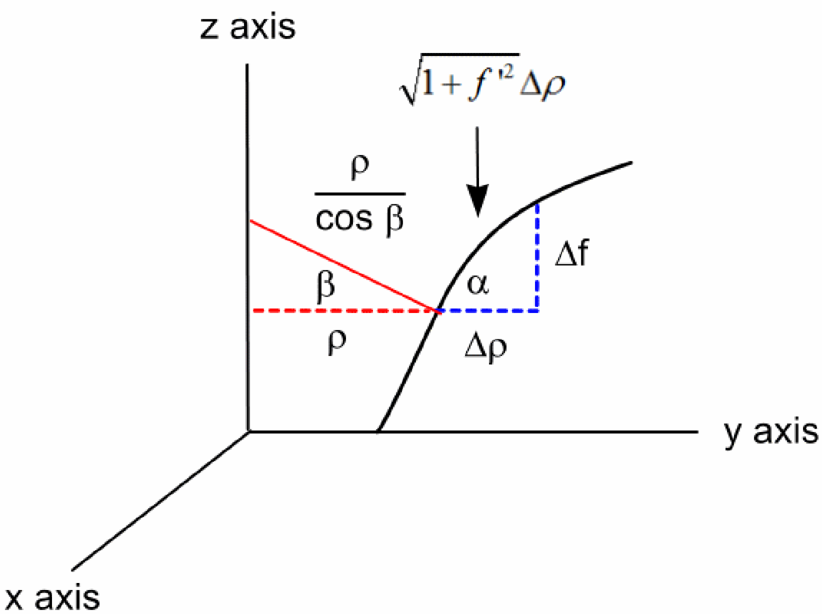
The infinitesimal length along the profile curve is $\sqrt{(1 + f'^2)}\,d\rho$ and the infinitesimal $\rho d\phi$ is simply the arc swept out by a radius ρ moving through an angle $d\phi$. The area of the patch is given by

$$dA = \sqrt{(1 + f'^2)}\,\rho\, d\phi\, d\rho \, .$$

Homework HW-14. Surface Areas. Find the surface areas for the following surfaces.



We have a definition for curvature for a one-dimensional curve: $K = \frac{|y''|}{(1+y'^2)^{3/2}} = \frac{1}{R}$. We need to extend this definition to a curvature for a surface.



Think of our two-dimensional surface as having two perpendicular curvatures:

1. the curvature for the profile:

$$K_1 = \frac{f''}{(1+f'^2)^{3/2}} = \frac{1}{R_1}$$

We dropped the absolute value sign to allow for positive and negative curvature – which we want for surfaces.

2. the curvature due to the revolution. We will call this curvature K_2.

We need to find the radius of curvature R_2 for the revolution. This radius must be perpendicular to the profile curve (to be fully independent of R_1) and swing into the page. Think of curving along the profile curve for K_1 and curving into the page for K_2. The two infinitesimal sweep segments are perpendicular to each other.

From the figure: $R_2 = \frac{\rho}{\cos\beta}$. As this hypotenuse swings around the z-axis, it traces out a cone. The curvature K_2 is the reciprocal:

$$K_2 = \frac{\cos\beta}{\rho}.$$

Since $\alpha + \beta = 90^\circ$, $K_2 = \frac{\sin\alpha}{\rho} = \frac{1}{\rho}\frac{\Delta f}{\sqrt{1+f'^2}\Delta\rho} \rightarrow \frac{1}{\rho}\frac{1}{\sqrt{1+f'^2}}\frac{df}{d\rho}$.

The two curvatures are

$$K_1 = \frac{f''}{(1+f'^2)^{3/2}} = \frac{1}{R_1} \quad \text{and} \quad K_2 = \frac{1}{\rho}\frac{f'}{\sqrt{1+f'^2}} = \frac{1}{R_2} .$$

We will define the curvature for the surface as the product:

$$K = K_1 K_2 = \frac{1}{R_1 R_2} = \frac{f''}{(1+f'^2)^{3/2}} \frac{1}{\rho}\frac{f'}{\sqrt{1+f'^2}} \text{, which}$$

simplifies to

$$K = \frac{f' f''}{\rho(1+f'^2)^2} .$$

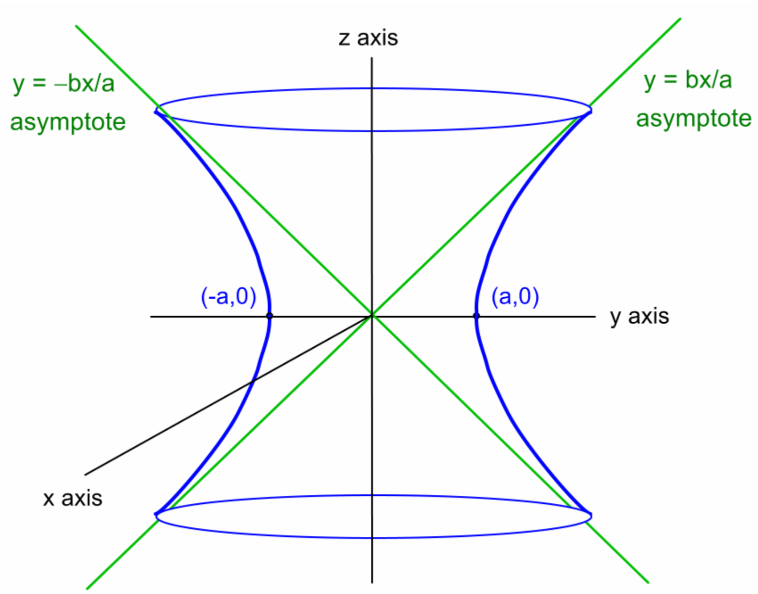
This curvature is also known as the Gaussian curvature.

Homework HW-15. Gaussian Curvature for a Sphere. Show that the Gaussian curvature for a sphere with radius R is $\frac{1}{R^2}$.

Homework HW-16. Gaussian Curvature for a Hyperboloid. Show that the Gaussian curvature for a hyperboloid of revolution with boundary profile

$$\left[\frac{\rho}{a}\right]^2 - \left[\frac{z}{b}\right]^2 = 1 \quad \text{equals} \quad K = -\frac{b^2 a^4}{\left[(a^2+b^2)\rho^2 - a^4\right]^2} .$$

The curvature is negative because the surface curves outward like a wine glass.



GR2-5. The Metric Tensor. The first fundamental form gives us a way of calculating the lengths of paths on a surface. The information contained in the form can be summarized by listing the coefficients of the coordinate infinitesimals. As an example, consider our two-dimensional surface of revolution.

$$ds^2 = (1 + f'^2)d\rho^2 + \rho^2 d\phi^2$$

The form is quadratic and it is sometimes referred to as the fundamental quadratic form of space. If we let $g_{11} = (1 + f'^2)$ and $g_{22} = \rho^2$, we can write

$$ds^2 = g_{11} d\rho^2 + g_{22} d\phi^2 .$$

The coefficients g_{11} and g_{22}, which refer to the first coordinate ρ and second coordinate ϕ respectively, can be placed in a matrix of the form

$$g_{ij} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix},$$

where i refers to the row and j the column. The entries g_{12} and g_{21} are coefficients for possible terms $d\rho d\phi$ and $d\phi d\rho$. These do not appear in our specific case of surfaces of revolution. Therefore, $g_{12} = g_{21} = 0$. The matrix is called the metric tensor. For our surfaces of revolution, the metric tensor is

$$g_{ij} = \begin{bmatrix} 1 + f'^2 & 0 \\ 0 & \rho^2 \end{bmatrix}.$$

Homework HW-17. Gaussian Curvature and Metric for Surfaces of Revolution. Show that for surfaces of revolution,

$$K = \frac{1}{2\rho g_{11}^2} \frac{dg_{11}}{d\rho} .$$

The components of the tensor, i.e., the entries in the matrix, are dependent on the coordinates chosen to describe the surface. However, the Gaussian curvature is independent of the choice of coordinates and is a measure of the intrinsic curvature of the surface. Similarly, ds^2 is also independent of the choice of coordinates and gives the intrinsic infinitesimal distance along the space. In three dimensions, two examples are Cartesian and spherical coordinates:

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 .$$

Now we demonstrate that g_{ij} is indeed a tensor and a covariant one at that, which will justify our use of subscripts rather than superscripts. In general, the square of the line element, allowing for possible cross terms, is

$$ds^2 = g_{ij} dx^i dx^j ,$$

where the Einstein summation convention is assumed. The invariance of the square of the line element is another way of saying that it does not matter what coordinate system you use. For a primed and unprimed set of coordinates we have

$$ds^2 = g_{ij} dx^i dx^j = g'_{kl}\, dx'^k\, dx'^l .$$

Note the use of different summation letters on the right. This prevents confusion as these indices are summed over independently on each side. Each index goes from 1 to 3 in a three-dimensional space.

Using the chain rule, $g_{ij} \dfrac{dx^i}{dx'^k} dx'^k \dfrac{dx^j}{dx'^l} dx'^l = g'_{kl}\, dx'^k\, dx'^l$

We next bring everything to the right side and factor out the common pieces.

$$\left[g'_{kl} - g_{ij} \frac{dx^i}{dx'^k} \frac{dx^j}{dx'^l} \right] dx'^k\, dx'^l = 0$$

Since the differentials are arbitrary as we can pick different paths in general,

$$g'_{kl} = \frac{dx^i}{dx'^k} \frac{dx^j}{dx'^l} g_{ij} .$$

This result is the transformation of a covariant tensor of Rank 2. It is also symmetric.

Cartesian Coordinates (x, y, z). The Cartesian metric is given below.

$$g_{i\,j} = \delta_{i\,j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Leopold Kronecker (1823-1891)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

The Kronecker Delta symbol is defined as

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

and named after the German mathematician Leopold Kronecker. It is a symmetric symbol.

Cylindrical Coordinates (ρ, ϕ, z). The metric for Cylindrical Coordinates .

$$g_{i\,j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Spherical Coordinates (r, θ, ϕ). The metric for Spherical Coordinates .

$$g_{i\,j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}$$

GR2-6. Raising and Lowering Indices. Watch this trick. Take a contravariant vector A^i. The contravariant transformation is $A'^i = \frac{\partial x'^i}{\partial x^j} A^j$. Now construct in a primed frame,

$$A'(k) = g'_{ki} A'^i,$$

where we put the k index in parenthesis because we do not know at this point where is should go (subscript or superscript). Substitute

$A'^i = \frac{\partial x'^i}{\partial x^j} A^j$ and $g'_{ki} = \frac{\partial x^n}{\partial x'^k} \frac{\partial x^m}{\partial x'^i} g_{nm}$ into the above equation to get

$$A'(k) = \frac{\partial x^n}{\partial x'^k} \frac{\partial x^m}{\partial x'^i} g_{nm} \frac{\partial x'^i}{\partial x^j} A^j .$$

Regroup the factors to obtain

$$A'(k) = \frac{\partial x^n}{\partial x'^k} \left[\frac{\partial x^m}{\partial x'^i} \frac{\partial x'^i}{\partial x^j} \right] g_{nm} A^j .$$

The bracketed part reduces to

$$\frac{\partial x^m}{\partial x'^i} \frac{\partial x'^i}{\partial x^j} = \frac{\partial x^m}{\partial x^j} = \delta_{mj}$$

using the chain rule and the fact that the coordinates are independent of each other. Think of these relations at play here:

$\frac{\partial x}{\partial x} = 1$, $\frac{\partial x}{\partial y} = 0$, $\frac{\partial x}{\partial z} = 0$ and so on.

Then $A'(k) = \frac{\partial x^n}{\partial x'^k}\left[\frac{\partial x^m}{\partial x'^i}\frac{\partial x'^i}{\partial x^j}\right] g_{nm} A^j = \frac{\partial x^n}{\partial x'^k}\delta_{mj} g_{nm} A^j = \frac{\partial x^n}{\partial x'^k} g_{nj} A^j$.

Since $A'(k) = \frac{\partial x^n}{\partial x'^k} g_{nj} A^j = \frac{\partial x^n}{\partial x'^k} A(n)$, we see that the A entity transforms as a covariant vector. So we should write a subscript for the A components.

$$A'_k = \frac{\partial x^n}{\partial x'^k} A_n .$$

We constructed a covariant vector from a contravariant one using the metric tensor.

$$A_i = g_{ij} A^j$$

We call this procedure lowering an index. What about raising an index?

Let $g^{-1}{}_{ij}$ be the inverse of g_{ij} so $g^{-1}{}_{ij} g_{jl} = \delta_{il}$. Note the sum over j as this is the rule for multiplying matrices. Then,

$$g^{-1}{}_{ki} A_i = g^{-1}{}_{ki} g_{ij} A^j = \delta_{kj} A^j = A^k .$$

We have raised the index, i.e., transformed a covariant vector into its contravariant counterpart.

Homework HW-18. Obtaining a Covariant Tensor of Rank 2. Prove that

$A_{ij} = g_{im} g_{jn} A^{mn}$ is a covariant tensor of Rank 2.

GR2-7. The Contravariant Metric Tensor. We will now show that $g^{-1}{}_{ij}$ is a contravariant tensor. Start with

$A'^l = \frac{\partial x'^l}{\partial x^k} A^k$, $A'^l = g'_{lj}{}^{-1} A'_j$, and $A^k = g_{ki}{}^{-1} A_i$.

Then $A'^l = \frac{\partial x'^l}{\partial x^k} A^k$ with the above substitutions for A'^l and A^k becomes

$$g'_{lj}{}^{-1} A'_j = \frac{\partial x'^l}{\partial x^k} g_{ki}{}^{-1} A_i .$$

Now substitute $A_i = \frac{\partial x'^j}{\partial x^i} A'_j$ to get $g'_{lj}{}^{-1} A'_j = \frac{\partial x'^l}{\partial x^k} g_{ki}{}^{-1} \frac{\partial x'^j}{\partial x^i} A'_j$.

Bringing everything to one side

$$\left[g'_{lj}{}^{-1} - \frac{\partial x'^l}{\partial x^k} g_{ki}{}^{-1} \frac{\partial x'^j}{\partial x^i} \right] A'_j = 0 .$$

Since the vector components are arbitrary, i.e., the equation must be true for all vector components, the term inside the brackets must vanish. You just can't count on the A' components to conspire to get 0. That would be a chance occurrence.

We come to

$$g'_{lj}{}^{-1} - \frac{\partial x'^l}{\partial x^k} g_{ki}{}^{-1} \frac{\partial x'^j}{\partial x^i} = 0$$

$$g'_{lj}{}^{-1} - \frac{\partial x'^l}{\partial x^k} \frac{\partial x'^j}{\partial x^i} g_{ki}{}^{-1} = 0$$

$$g'_{lj}{}^{-1} = \frac{\partial x'^l}{\partial x^k} \frac{\partial x'^j}{\partial x^i} g_{ki}{}^{-1}$$

The last equation indicates that the inverse g transforms as a contravariant tensor of Rank 2. Therefore, we can write

$$g_{lj}{}^{-1} = g^{lj} \quad \text{and} \quad g'^{lj} = \frac{\partial x'^l}{\partial x^k} \frac{\partial x'^j}{\partial x^i} g^{ki} .$$

We have the contravariant metric tensor. It is the inverse of the covariant metric tensor. For Cartesian coordinates, everything is super simple:

$$g_{ij} = \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad g^{ij} = \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Homework HW-19. Contravariant Metric Tensor for Spherical Coordinates. Show by explicit calculation of

$$g^{ij} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} \delta^{kl} \text{, where}$$

δ^{kl} refers to the Cartesian coordinates and g^{ij} refers to spherical coordinates, that

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{r^2} & 0 \\ 0 & 0 & \dfrac{1}{r^2 \sin^2\theta} \end{bmatrix}.$$

I like to write the g with indices i and j. You can pick any index letters you want.

GR2-8. Contraction. What are invariant quantities, i.e., quantities that are independent of which coordinates we choose to use? One example is the square of the arc length?

$$ds^2 = g_{ij} dx^i dx^j$$

Another example, inspired by the above, is

$$A^2 \equiv g_{ij} A^i A^j .$$

This equation is equivalent to

$$A^2 = A_i A^i \text{ since } A_i = g_{ij} A^j .$$

The above formulation is our dot product for a vector $\vec{A} \cdot \vec{A} = A_i A^i$. Note that

$$A^2 = A_i A^i = A'_j A'^j \text{. The length is invariant.}$$

In Cartesian coordinates $A_i = g_{ij} A^j = \delta_{ij} A^j = A^i$. So we can write components as subscripts always in Cartesian systems like you do in physics class. In the Cartesian case we find

$$\vec{A} \cdot \vec{A} = A_i A^i = A_i A_i = A_x A_x + A_y A_y + A_z A_z = A_x^2 + A_y^2 + A_z^2 .$$

Homework HW-20. Constructing Tensors. Show $T^{ij} = A^i A^j$ is a contravariant tensor of Rank 2, given that A^i is a contravariant vector. Then form $T_k{}^j = g_{ki} T^{ij}$. Show that $T_j{}^j$ is an invariant.

What you did in HW-20 is called contraction – you make a covariant index the same as a contravariant one as in $T_j{}^j$. You can make scalar invariants this way. But be careful to follow two rules:

1. always pair a covariant index with a contravariant one,
2. you must pair up all the indices so that you are left with a scalar.

As another example, taking $T^{ij}{}_{nm}$ and doing the contraction $T^{ij}{}_{ij}$ we wind up with a scalar. But $T^{ii}{}_{jj}$ is NOT a scalar. We did not pair a contravariant index with a covariant one. Note that tensors of odd rank cannot be reduced to scalars because there is always an odd-one out that cannot be paired.

Personal Note. When I was an undergraduate, I enjoyed the Vector Analysis book in the Schaum's Outline Series. But there was no Tensor book in the series then. The Tensor Calculus book was finally published in 1988 by David Kay, who had come to UNC Asheville as the Chair of Mathematics in the 1980s. At that time I was Chair of Physics. Dr. Kay formerly taught in the graduate program at the University of Oklahoma for 17 years.

Dr. Kay has retired since. He paid us a visit in 2011, the year that the revised edition of his book appeared in the Schaum's Outline Series. See him in the above photo with your instructor (2011).