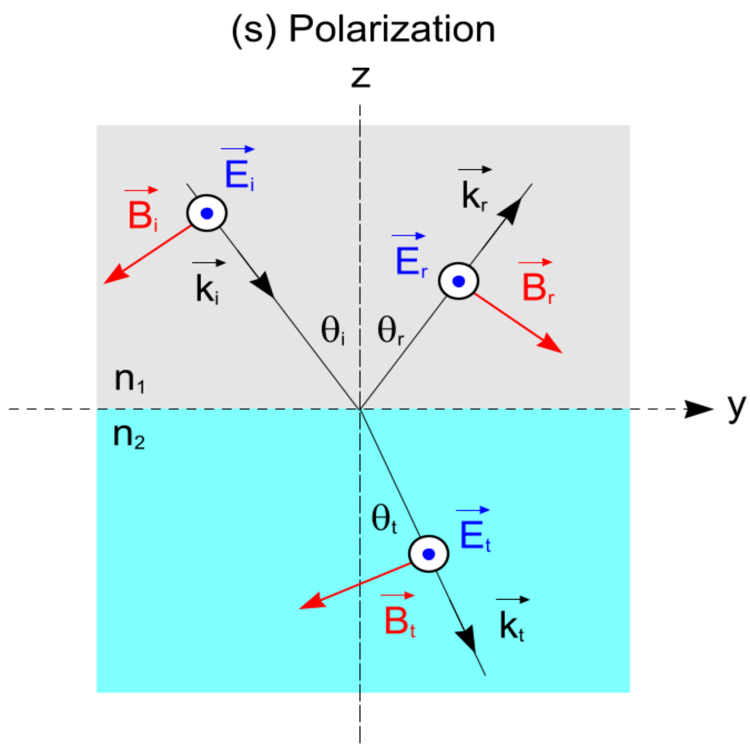
**Modern Optics, Prof. Ruiz, UNCA** doctorphys.com
**Chapter N. Fresnel Equations – Solutions**

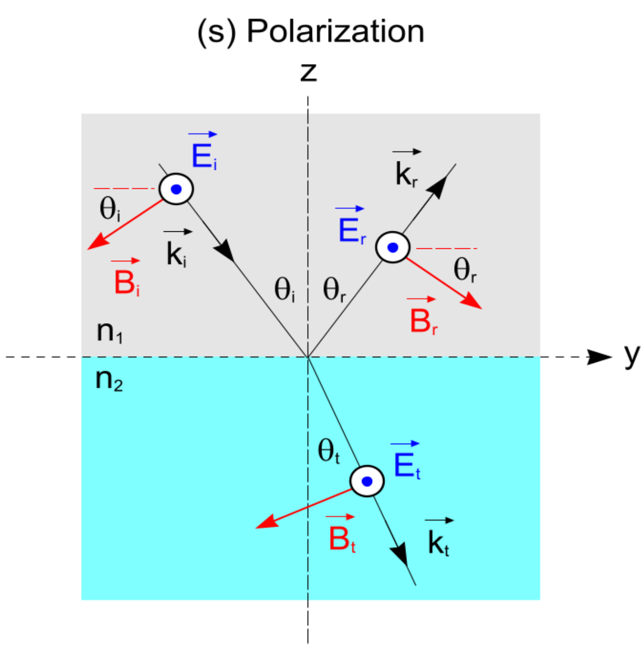
**HW N1. Polarization (s) Theoretical Derivation.** We derived the Fresnel equations for the p-polarization case in class and found $r_p$ and $t_p$. Derive the Fresnel equations for the s-polarization shown in the figure. The answers are given below.

$$r_s = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}$$

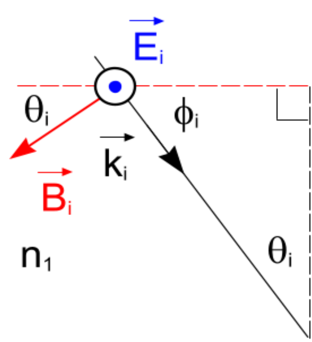
$$t_s = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2}$$

Be particular sure to explain your angles in the analysis of the boundary conditions at the interface, working out one in detail.




**Solution.** First explain angles. The angles of the **B** vectors with respect to the horizontal are the same as their respective angles of incidence, reflection, and transmission. One is worked out in detail below.




The angle made with $\mathbf{B}_i$ and the horizontal is the same as the angle of incidence due to both of these angles being complementary to $\phi_i$.

Therefore, for the tangential components of the magnetic field to match at the boundary,

$$-B_{i0}\cos\theta_i + B_{r0}\cos\theta_r = -B_{t0}\cos\theta_t ,$$

where minus signs mean that the **B** vector component is in the negative y direction.

For the tangential components of the electric field to match at the boundary,

$E_{i0} + E_{r0} = E_{t0}$ as all point out of the page in our figure (same direction).

But $\theta_i = \theta_r$ from the law of reflection. So let's call these angles $\theta_1$, angles in the first medium.

Then let's call $\theta_t$, the angle in the second medium, $\theta_2$.

Our equations become

$E_{i0} + E_{r0} = E_{t0}$ and $-B_{i0}\cos\theta_1 + B_{r0}\cos\theta_1 = -B_{t0}\cos\theta_2$, i.e.,

$E_{i0} + E_{r0} = E_{t0}$ and $\cos\theta_1(B_{i0} - B_{r0}) = B_{t0}\cos\theta_2$.

Now we recall the connecting equation $\dfrac{E_o}{B_o} = c$ in vacuum and $\dfrac{E_o}{B_o} = v$ in general with

$n = \dfrac{c}{v}$. Therefore, $\dfrac{B_o}{E_o} = v = \dfrac{n}{c}$ and $B_o = \dfrac{n}{c}E_o$. Then

$\cos\theta_1(B_{r0} - B_{i0}) = B_{t0}\cos\theta_2$ becomes $\cos\theta_1\left(\dfrac{n_1 E_{i0}}{c} - \dfrac{n_1 E_{r0}}{c}\right) = \dfrac{n_2 E_{t0}}{c}\cos\theta_2$,

or simply $\cos\theta_1(E_{i0} - E_{r0}) = E_{t0}\cos\theta_2$.

Summary: $E_{i0} + E_{r0} = E_{t0}$ and $n_1\cos\theta_1(E_{i0} - E_{r0}) = n_2 E_{t0}\cos\theta_2$.

Divide our equations by $E_{i0}$ to get

$1 + \dfrac{E_{r0}}{E_{i0}} = \dfrac{E_{t0}}{E_{i0}}$ and $n_1\cos\theta_1\left(1 - \dfrac{E_{r0}}{E_{i0}}\right) = n_2\dfrac{E_{t0}}{E_{i0}}\cos\theta_2$.

With our usual definitions $r = \frac{E_{r0}}{E_{io}}$ and $t = \frac{E_{t0}}{E_{io}}$,

$1 + r = t$ and $n_1 \cos\theta_1 (1 - r) = n_2 t \cos\theta_2$.

$$n_1 \cos\theta_1 (1 - r) = n_2 (1 + r) \cos\theta_2$$

$$n_1 \cos\theta_1 - n_1 r \cos\theta_1 = n_2 \cos\theta_2 + n_2 r \cos\theta_2$$

$$n_1 \cos\theta_1 - n_2 \cos\theta_2 = n_1 \cos\theta_1 r + n_2 r \cos\theta_2$$

$$\boxed{r = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}}$$

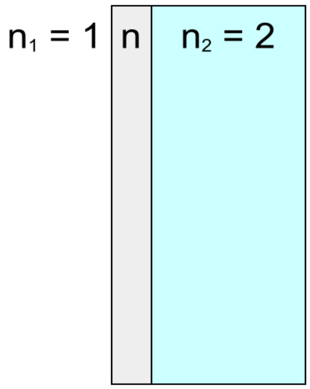
$$t = 1 + r = 1 + \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}$$

$$t = \frac{(n_1 \cos\theta_1 + n_2 \cos\theta_2) + (n_1 \cos\theta_1 - n_2 \cos\theta_2)}{n_1 \cos\theta_1 + n_2 \cos\theta_2}$$

$$\boxed{t = \frac{2 n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2}}$$

For the "Final Answers" we add the subscript "s" to each.

$$\boxed{r_s = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}} \qquad \boxed{t_s = \frac{2 n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2}}$$




**HW N2. Thin-Film Engineering.**

A manufacturing firm is designing a transparent plate with index of refraction $n_2 = 2.00$. They would like you to design a thin film with index of refraction n so that light entering from the air will have as little reflection back into the air as possible and transmission to the $n_2$ material is maximized.

Use the Fresnel equations at normal incidence to minimize the reflectivity. Remember that the reflectivity $R_p$ is equal to the square of the reflection coefficient $r_p$ of the Fresnel equations, that $R_s$ is the square of $r_s$, and

$$R = (R_p + R_s) / 2.$$

The index of refraction of air is $n_1 = 1.00$ to 3 significant figures and you are given $n_2 = 2.00$ to 3 significant figures. Note that there are two R interfaces: reflection at the $n_1$-n interface and reflection at the n-$n_2$ interface. You will need to incorporate these in your analysis as well as worry about $R_p$ and $R_s$. Report your value n for the thin film to 3 significant figures.

**Solution.** The Fresnel equations for the reflection coefficients are

$$r_p = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_1 \cos\theta_2 + n_2 \cos\theta_1} \qquad r_s = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}$$

For normal incidence,

$$r_p = \frac{n_2 \cos 0^\circ - n_1 \cos 0^\circ}{n_1 \cos 0^\circ + n_2 \cos 0^\circ} = \frac{n_2 - n_1}{n_1 + n_2} \qquad r_s = \frac{n_1 - n_2}{n_1 + n_2}$$

$$R_p = r_p^2 = \left[\frac{n_2 - n_1}{n_1 + n_2}\right]^2 \qquad R_s = r_s^2 = \left[\frac{n_1 - n_2}{n_1 + n_2}\right]^2 = R_p$$

$$R = \frac{R_p + R_s}{2} = \left[\frac{n_1 - n_2}{n_1 + n_2}\right]^2$$

But there are two relevant interfaces where reflection occurs.

We have $R_1 = \left[\frac{1-n}{1+n}\right]^2 = \left[\frac{n-1}{n+1}\right]^2$ and $R_2 = \left[\frac{n-2}{n+2}\right]^2$.

$$T_1 = 1 - R_1 = 1 - \left[\frac{n-1}{n+1}\right]^2 = \frac{(n+1)^2 - (n-1)^2}{(n+1)^2}$$

$$T_1 = \frac{(n^2 + 2n + 1) - (n^2 - 2n + 1)}{(n+1)^2} = \frac{4n}{(n+1)^2}$$

$$T_2 = 1 - R_1 = 1 - \left[\frac{n-2}{n+2}\right]^2 = \frac{(n+2)^2 - (n-2)}{(n+2)^2}$$

$$T_1 = \frac{(n^2 + 4n + 4) - (n^2 - 4n + 4)}{(n+1)^2} = \frac{8n}{(n+1)^2}$$

We need to find n that minimizes $T_{\text{total}} = \frac{4n}{(n+1)^2} \frac{8n}{(n+2)^2} I$ .

The procedure is a max-in problem: $\frac{dT_{\text{total}}}{dn} = 0$

$$\frac{d}{dn}\left[ n^2 (n+1)^{-2} (n+2)^{-2} \right] = 0$$

Note that if $\frac{d}{dx}(fgh) = f'gh + fg'h + fgh' = 0$, then divide by $fgh$ to get

$$\frac{1}{fgh}\frac{d}{dx}(fgh) = \frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h} = 0$$

Let $f = n^2$, $g = (n+1)^{-2}$, and $h = (n+2)^{-2}$. Then

$$\frac{1}{n^2 (n+1)^{-2} (n+2)^{-2}} \frac{d}{dn}\left[ n^2 (n+1)^{-2} (n+2)^{-2} \right] = 0$$ and

$$\frac{2n}{n^2} - 2\frac{(n+1)^{-3}}{(n+1)^{-2}} - 2\frac{(n+2)^{-3}}{(n+2)^{-2}} = 0$$

$$\frac{1}{n} - \frac{1}{(n+1)} - \frac{1}{(n+2)} = 0$$

$$\frac{(n+1)(n+2) - n(n+2) - n(n+1)}{n(n+1)(n+2)} = 0$$

The numerator will be zero.

$$(n+1)(n+2) - n(n+2) - n(n+1) = 0$$

$$(n^2 + 3n + 2) - n^2 - 2n - n^2 - n = 0$$

$$(n^2 - n^2 - n^2) + (3n - 2n - n) + 2 = 0$$

$$-n^2 + 0 + 2 = 0$$

$$n^2 = 2$$

$$\boxed{n = \sqrt{2}} \qquad \boxed{n = 1.41}$$

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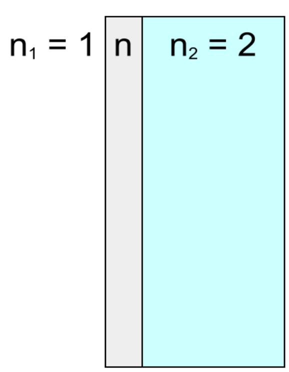


Extra General Comment

In general: $R_1 = \left[\frac{n_1 - n}{n_1 + n}\right]^2$ and $R_2 = \left[\frac{n - n_2}{n + n_2}\right]^2$. You get for the maximum transmission the geometric mean.

$$\boxed{n = \sqrt{n_1 n_2}}$$

For our specific case $n_1 = 1$ and $n_2 = 2$, giving $n = \sqrt{n_1 n_2} = \sqrt{1 \cdot 2} = \sqrt{2}$.