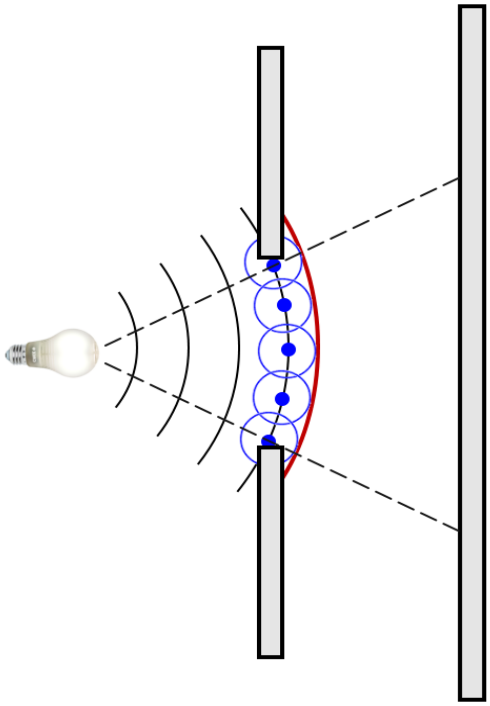
U1. The Fresnel-Kirchhoff Framework.



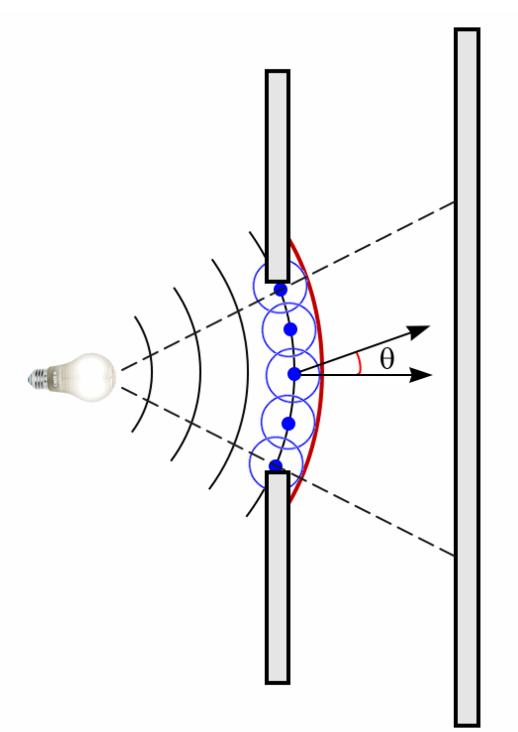
Spherical waves travel outward from a source indicated by the light bulb. According to the Huygens-Fresnel principle, any spherical wave can be considered to consist of tiny secondary wavelets, the baby waves. See the blue secondary wavelets in the figure.

The red crest is the result of the outgoing "baby" wavelets. Since shadows are fairly sharp, the wavelets must interfere destructively at the extreme angles. Otherwise, you would not get the proper shadows due to the two vertical walls that border the opening.

Also, the wavelets cannot be uniform in strength in all directions. After all, you cannot have baby waves traveling back to the light bulb.

Therefore, in this model, we can at least say that the backward waves must be suppressed. The obliquity factor

$$\frac{(1+\cos\theta)}{2}$$



for the baby waves is the simplest form to do the trick. In the forward direction $\theta = 0°$ and

$$\frac{(1+\cos 0°)}{2} = \frac{1+1}{2} = 1 \ .$$

In the backward direction $\theta = 180°$ and

$$\frac{(1+\cos 180°)}{2} = \frac{1-1}{2} = 0 \ .$$

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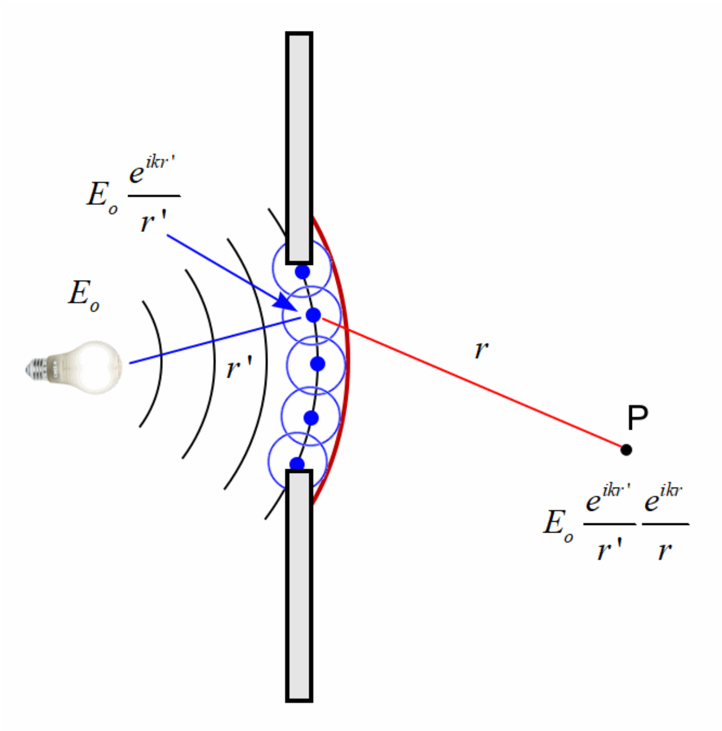

Let the source strength be E_o. The strength at distance r' from the source, somewhere along the aperture is then $E_o \frac{e^{ikr'}}{r'}$. The strength beyond the aperture at a distance r' introduces a baby wave factor:

$$E_o \frac{e^{ikr'}}{r'} \frac{e^{ikr}}{r}.$$

We want to integrate over the aperture to get the total strength at P:

$$E_p = \int_{Aperture} E_o \frac{e^{ikr'}}{r'} \frac{e^{ikr}}{r} dA$$

Now throw in the obliquity factor:

$$E_p = \int_{Aperture} E_o \frac{(1+\cos\theta)}{2} \frac{e^{ikr'}}{r'} \frac{e^{ikr}}{r} dA$$

This equation is a simplified form of the Fresnel-Kirchhoff diffraction equation. For Fraunhofer diffraction the point P is very far away and the angle spread for θ is very small. Therefore

$$\frac{(1+\cos\theta)}{2} \to 1.$$

We analyzed cases of uniform light passing through the aperture, i.e., $\frac{e^{ikr'}}{r'} = \text{const}$. Then, in the far field of the Fraunhofer realm,

$$E_p = \int_{Aperture} E_o \frac{(1+\cos\theta)}{2} \frac{e^{ikr'}}{r'} \frac{e^{ikr}}{r} dA \to \frac{e^{ikr'}}{r'} \int_{Aperture} E_o \frac{e^{ikr}}{r} dA.$$

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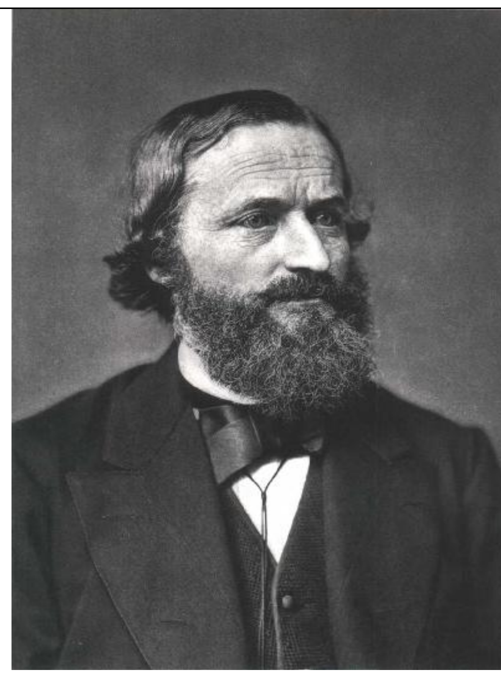
Absorbing the constant $\frac{e^{ikr'}}{r'}$ into E_o, we therefore have the starting point of our Fraunhofer cases in the previous chapters

$$E_p = \int_{Aperture} E_o \frac{e^{ikr}}{r} dA .$$

But the more general diffraction formula is

$$E_p = \int_{Aperture} E_o \frac{(1+\cos\theta)}{2} \frac{e^{ikr'}}{r'} \frac{e^{ikr}}{r} dA .$$

 Augustin-Jean Fresnel	 Gustave Robert Kirchhoff
1788 - 1827	1824 - 1887
Civil Engineer and Physicist	Physicist

Remember Kirchhoff's laws in circuits?

Kirchhoff's Junction Law: The algebraic sum of currents at a junction is zero.
Kirchhoff's Voltage Law: The voltages around any closed loop add to zero.

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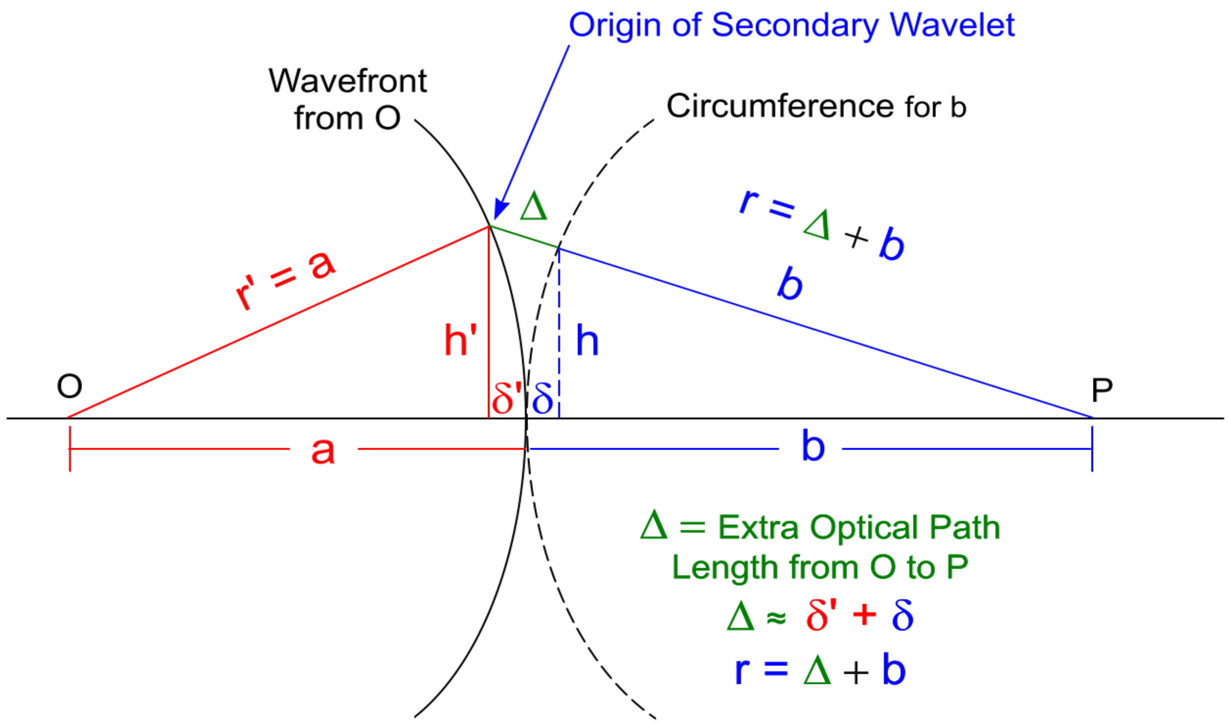
U2. Wavefront and Optical Path.



$$\delta = b - \sqrt{b^2 - h^2} \quad \Rightarrow \quad \delta = b - b\left[1 - \left(\frac{h}{b}\right)^2\right]^{1/2}$$

$$\delta \approx b - b\left[1 - \frac{1}{2}\left(\frac{h}{b}\right)^2\right] = \frac{h^2}{2b}$$

Similarly $\delta' \approx \dfrac{h'^2}{2a}$.

$$\Delta = \delta' + \delta \quad \Rightarrow \quad \Delta = \frac{h'^2}{2a} + \frac{h^2}{2b}$$

From crests near the horizontal axis, $h \approx h'$.

$$\boxed{\Delta = \frac{h^2}{2}\left(\frac{1}{a} + \frac{1}{b}\right)}$$

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For Fraunhofer diffraction

$$\frac{e^{ikr'}}{r'} = \text{const} ,$$

which means that the waves reaching the aperture are virtually plane waves. Therefore, Δ has to be very, very small compared to the wavelength. Another way of looking at things is that the distance "a" has to be very large in order to get the plane waves. Also note that large r on the other side of the aperture indicates smaller Δ values due the small 1/r = 1/b in its formula.

$$\Delta = \frac{h^2}{2}(\frac{1}{a} + \frac{1}{b}) << \lambda$$

Consider a slit width to estimate h. A typical grating has 250 lines per mm. That means the distance between each slit is about

$$\frac{1 \text{ mm}}{250} = \frac{1000 \text{ microns}}{250} = 4 \text{ microns} .$$

Each slit width must be less that this distance. Let's take 1 micron for a single slit. Let r' = a = 20 cm be not so large for the source and r = b = 100 cm for the screen distance.

$$\Delta = \frac{h^2}{2}(\frac{1}{a} + \frac{1}{b}) \approx \frac{(10^{-6})^2}{2}(\frac{1}{1/5} + \frac{1}{1})$$

$$\Delta = \frac{(10^{-6})^2}{2}(5+1) = 3 \cdot 10^{-12} \text{ m} = 3 \cdot 10^{-3} \text{ nm} << 550 \text{ nm}$$

The middle of the visible spectrum has wavelength 550 nm.

Far-Field Diffraction --- Fraunhofer Diffraction

Near-Field Diffraction --- Fresnel Diffraction

Extreme-Near-Field --- Blinding Light through the Aperture --- "In Your Face"

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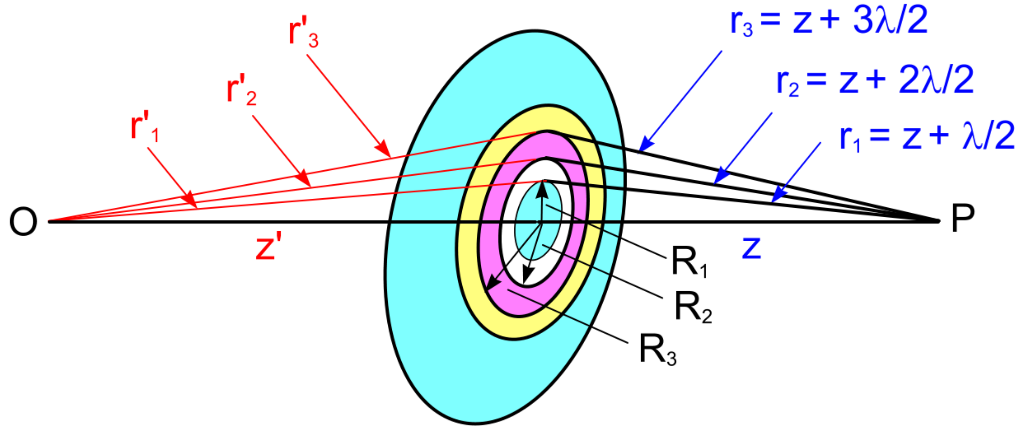
U3. Fresnel Zones (The Near Field).



Return to the extra path length formula $\Delta = \frac{h^2}{2}(\frac{1}{a} + \frac{1}{b})$ where from the figure $h = R$ (one of the radii), $a = z'$, and $b = z$. The figure shows concentric regions of the wavefront where the extra path length is given by additional half wavelengths:

$$\Delta_m = \frac{R_m^2}{2}(\frac{1}{z'} + \frac{1}{z}) = m\frac{\lambda}{2}$$, where m = 1, 2, 3, and so on.

$$E_p = \int_{Aperture} E_o \frac{(1+\cos\theta)}{2} \frac{e^{ikr'}}{r'} \frac{e^{ikr}}{r} dA$$

Fro small R we can take $r' \to z'$ and $r \to z$.

$$E_p = \frac{1}{zz'} \int_A E_o \frac{(1+\cos\theta)}{2} e^{ik(r'+r)} dA$$

Also for small R, we can take θ ≈ 0 and neglect the obliquity factor. Then.

$$E_p = \frac{E_o}{zz'} \int_A e^{ik(r'+r)} dA.$$

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Consider a general zone with r', r, and R. Then

$$r' + r = \sqrt{z'^2 + R^2} + \sqrt{z^2 + R^2}$$

$$r' + r = z'(1 + \frac{R^2}{z'^2})^{1/2} + z(1 + \frac{R^2}{z^2})^{1/2}$$

$$r' + r = z'(1 + \frac{R^2}{2z'^2}) + z\,(1 + \frac{R^2}{2z^2})$$

$$r' + r = z' + z + \frac{R^2}{2}(\frac{1}{z'} + \frac{1}{z})$$

Substitute $\Delta = \frac{R^2}{2}(\frac{1}{z'} + \frac{1}{z})$ from before to simplify notation.

$$e^{ik(r'+r)} = e^{ik(z'+z+\Delta)} = e^{ik(z'+z)} e^{ik\Delta}$$

$$E_p = \frac{E_o}{zz'} \int_A e^{ik(r'+r)}\, dA = \frac{E_o}{zz'} \int_A e^{ik(z'+z)} e^{ik\Delta}\, dA$$

$$E_p = \frac{E_o}{zz'} e^{ik(z'+z)} \int_A e^{ik\Delta}\, dA$$

$$\Delta_m = \frac{R_m^2}{2}(\frac{1}{z'} + \frac{1}{z}) = m\frac{\lambda}{2}$$

Define $\frac{1}{L} \equiv (\frac{1}{z'} + \frac{1}{z}) = \frac{z + z'}{z'z}$. Then $L = \frac{z'z}{z + z'}$

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$$\Delta_m = \frac{R_m^2}{2}(\frac{1}{z'} + \frac{1}{z}) = \frac{R_m^2}{2}\frac{1}{L} = m\frac{\lambda}{2} \quad => \quad R_m^2 = m\lambda L$$

$$R_m = \sqrt{m\lambda L}$$

The mth Fresnel zone has area

$$\pi R_{m+1}^2 - \pi R_m^2 = \pi(m+1)\lambda L - \pi m\lambda L = \pi\lambda L = \pi R_1^2$$

All the Fresnel zones have equal area in this approximation. Using $R_m^2 = m\lambda L$, the number of Fresnel zones for a circular aperture with radius R is $m = \frac{R_m^2}{\lambda L}$, which we call

the Fresnel number $F = \frac{R^2}{\lambda L}$.

What is the Fresnel number if $R = 1 \text{ cm}$, $z' = z = 10 \text{ cm}$, and $\lambda = 500 \text{ nm}$.

$$L = \frac{z'z}{z+z'} = \frac{10 \cdot 10}{10+10} = \frac{100}{20} = 5 \text{ cm}.$$

$$F = \frac{R^2}{\lambda L} = \frac{(10^{-2} \text{ m})^2}{(500 \times 10^{-9} \text{ m})(5 \times 10^{-2} \text{m})}$$

$$F = \frac{10^{-4}}{2500 \times 10^{-11}} = \frac{10^{-4}}{25 \times 10^{-9}} = \frac{1}{25} \times 10^5 = \frac{100,000}{25} = 4000$$

What is the radius of the central Fresnel zone for this case?

$$R_m = \sqrt{m\lambda L} \quad => \quad R_1 = \sqrt{\lambda L}$$

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$$R_1 = \sqrt{(500\times10^{-9}\ \text{m})(5\times10^{-2}\text{m})} = \sqrt{2500\times10^{-11}} = \sqrt{250\times10^{-10}}$$

To two sig figs: $R_1 = \sqrt{256\times10^{-10}} = 16\cdot10^{-5} = 0.16\cdot10^{-3} = 0.16\ \text{mm}$.

Consider by comparison a diffraction grating for Fraunhofer diffraction.

A grating can easily have 1000 lines per mm.

So take $R \sim 1\ \mu\text{m}$ and for far field take $L \sim 1\ \text{m}$.

$$F = \frac{R^2}{\lambda L} = \frac{(10^{-6})^2}{(500\times10^{-9})(1)} = \frac{1}{5}\times\frac{10^{-12}}{10^{7}} = \frac{10^{-5}}{5} << 1$$

Another way of saying this result of $\frac{R^2}{\lambda L} << 1$ is

$\frac{R^2}{L} << \lambda$, consistent with our earlier result that $\Delta << 550\ \text{nm}$.

U4. The Unobstructed Wave. Amplitudes at point P will be the sum of the secondary wavelets coming from the Fresnel zones. Note that the half wavelength added at each zone alternates phases by π. The amplitudes will alternate being in phase and out of phase.

$$E_p = E_1 - E_2 + E_3 - E_4 + E_5 - ...$$

Regroup these so that we have

$$E_p = \frac{E_1}{2} + (\frac{E_2}{2} - E_2 + \frac{E_3}{2}) + (\frac{E_3}{2} - E_4 + \frac{E_5}{2}) + ...$$

The middle one will be approximately the average of the neighbors so that

$E_p \approx \frac{E_1}{2}$, where we do not worry about the last one, which will be much weaker.

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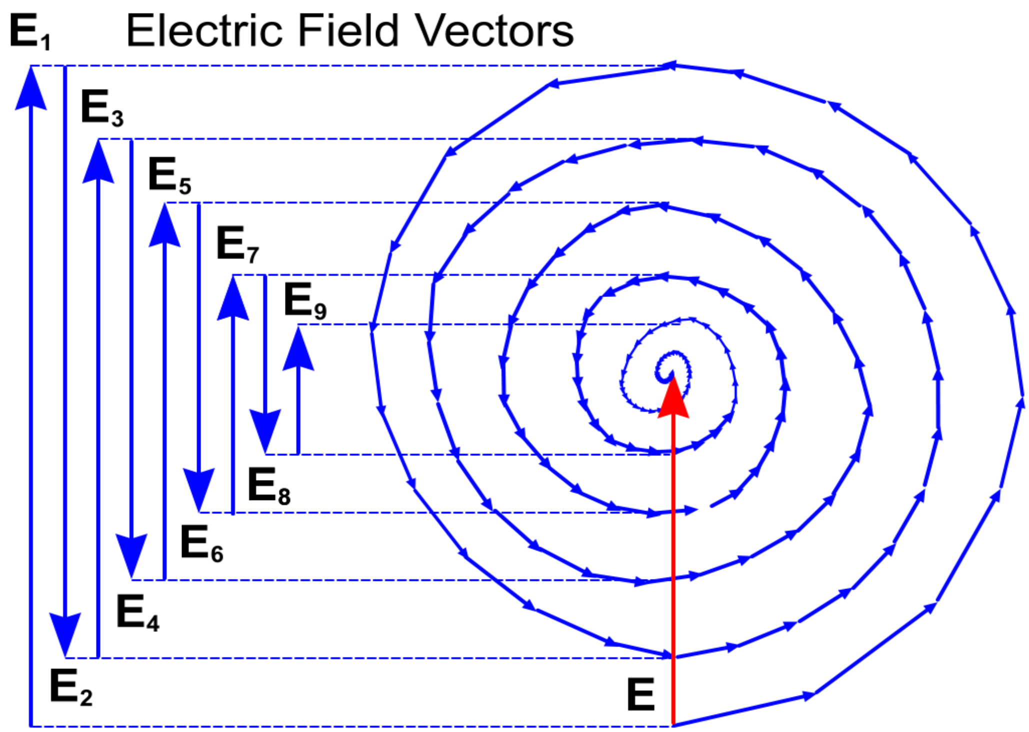
The unobstructed wave along the axis will have an irradiance

$$I = \frac{1}{2}\left|\frac{E_1}{2}\right|^2 = \frac{1}{4}\frac{1}{2}\left|E_1\right|^2$$, i.e., 25% of the strength for the first zone.

Vibration Curve: Fresnel Zone Electric Field Vectors



Consider each phasor arrow as coming from a subsection of a Fresnel zone.

From the figure you can see that the final sum vector is about half the length of the first zone vector, as we indicated earlier with our equation $E_p \approx \frac{E_1}{2}$. The above figure gives a nice visualization of the combinations of phasors to get each of the electric fields for each Fresnel zone: $E_p = E_1 - E_2 + E_3 - E_4 + E_5 - \cdots$. In vector form, the sum vector for all the Fresnel zones, as seen graphically in the figure called a **vibration curve**, is

$\overrightarrow{E_p} = \vec{E} = \frac{\overrightarrow{E_1}}{2}$ and the unobstructed irradiance is $I_o = \frac{1}{2}\left|\frac{E_1}{2}\right|^2 = \frac{1}{8}\left|E_1\right|^2$.

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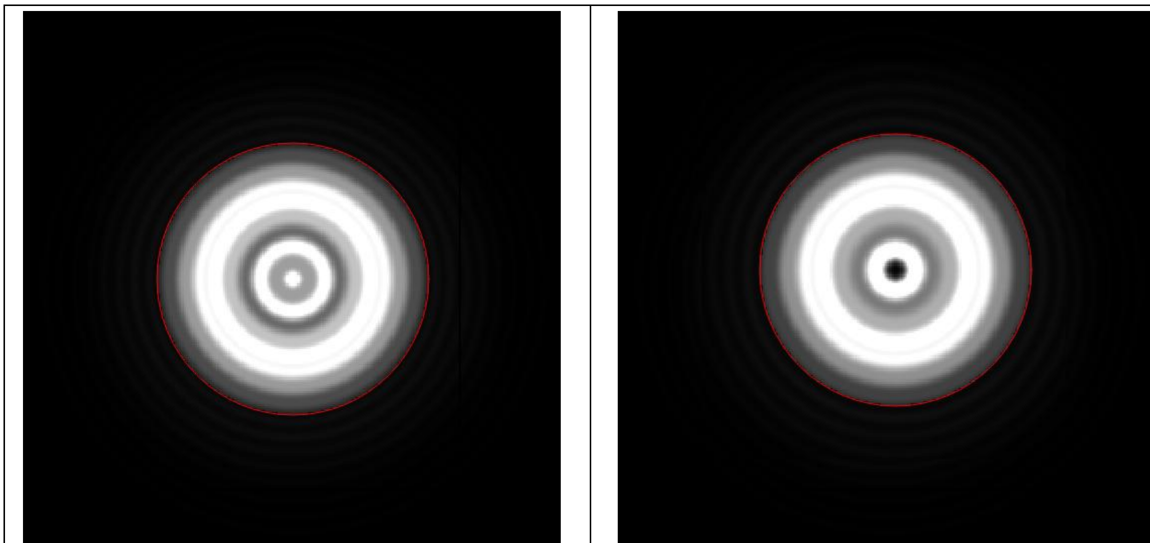
U5. Circular Aperture. If you place an aperture that blocks off all the zones except the first one, then at point P along the axis,

$$E_p = E_1 \quad \text{and} \quad I = \frac{1}{2}\left|E_1\right|^2, \quad \text{4 times as bright as the unobstructed wave.}$$

But if the aperture lets the first two zones through, we have destructive interference due to the extra $\lambda / 2$ optical path length from the second zone. You will get basically darkness. As you increase the radius, if an odd number of zones gets through, you have the bright spot about 4 times as great as the unobstructed wave. If the aperture is such that an even number of zones gets through, you have destructive interference with its corresponding dark spot along the central axis.

But there is more! If you move the point P along the axis towards or away from the aperture, then the sizes of the zones change since the distance r changes. So you get alternating bright and dark effects this way too.

Computer Generation of Fresnel Diffraction through a Circular Aperture



Circular Aperture with Bright Center	Circular Aperture with Bright Center

Courtesy www.falstad.com with credits to Bob Hanson and his JavaScript team.

Check out this video by David Velasco Villamizer which is an animation of the diffraction pattern seen as the point P moves away from the near-field (Fresnel diffraction) to the far-field (Fraunhofer diffraction). First there is blinding brightness when you are so close to the source. Then you see the alternating bright-dark central spots, which eventually settle down to Fraunhofer diffraction in the far field.

Diffraction Video by David Velasco Villamizer

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U6. Circular Obstacle. If you place a small circular obstacle or disk in front of the source, you block out inner zones. But the outer zones are still there. Consider a case where first, we have a bright spot somewhere along the axis. Then consider an obstacle that blocks out the first zone only. The contribution E_1 of the first zone is removed from the total $E_p \approx \frac{E_1}{2}$ of the unobstructed wave, giving

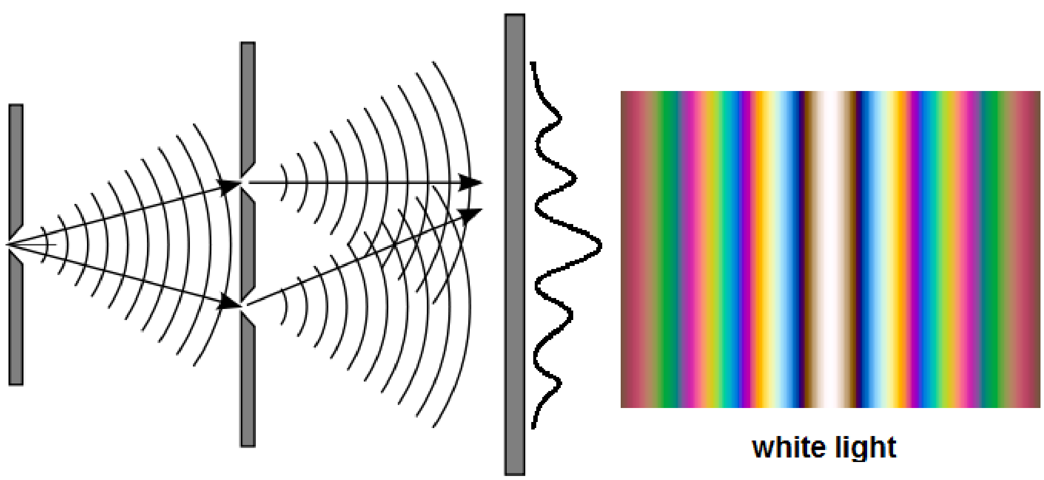
$$E_p \rightarrow \frac{E_1}{2} - E_1 = -\frac{E_1}{2}$$, same magnitude with a phase change.

Since $I = \frac{1}{2}\left|-\frac{E_1}{2}\right|^2 = \frac{1}{8}\left|E_1\right|^2$, you get the same bright spot as the unobstructed wave!

Of course, if you start moving along the axis, this will change and it can vanish.

Historically, this problem led to a show down that we describe next!

Historical Background. Light was controversial around 1800. Many thought light consisted of particles. Thomas Young in 1801 set up an experiment where light was passed through two small slits and an interference pattern was seen on a screen. Back in those days they did not have lasers. Here is what the two-slit interference pattern looks like with white light



Young's Two-slit Experiment

Wikipedia: Stannered, Epzcaw, and Stigmatella aurantiaca. Creative Commons

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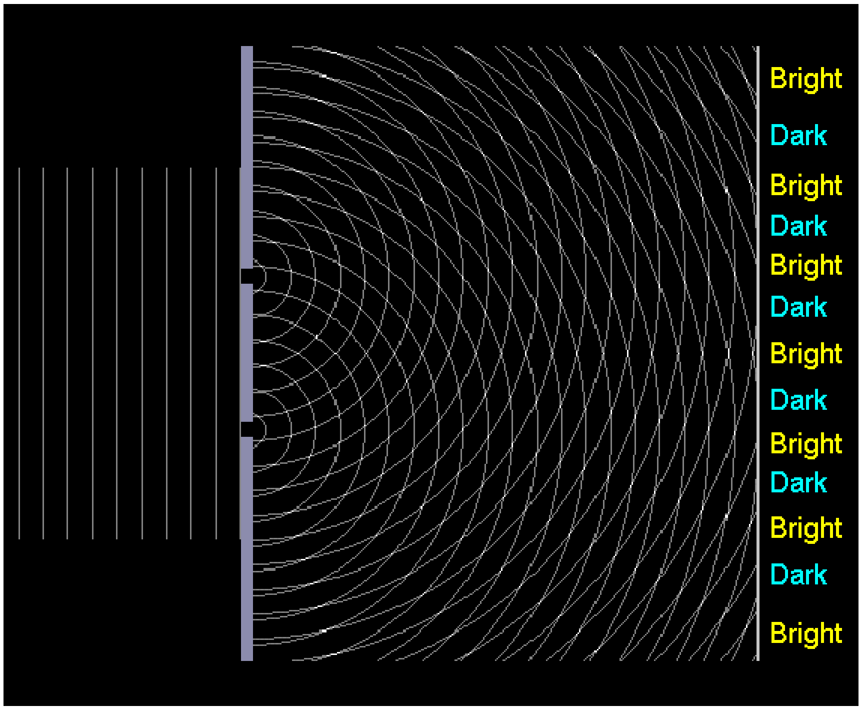
Simplified Two-Slit Interference Schematic



Remember our advanced treatment of this problem led to $I(\theta) = I(0)\cos^2\alpha \frac{\sin^2\beta}{\beta^2}$.

We might consider this experiment as the solid beginnings of the wave theory of light, i.e. physical optics!

The interference of light was discovered in 1801 by Thomas Young (1773-1829), an excellent example of an interdisciplinary scholar. Young was an physician and physicist, with knowledge of several languages. Young's experimental arrangement in discovering interference is essentially what we have described above - sending light through two closely-spaced slits. In fact, we refer today to the experiment as ***Young's Double Slit Experiment***.

His observation resurrected the wave model of light, which had fallen out of favor by scientists for 100 years since Newton argued for the particle model of light. Today we know that both models are needed to understand the different aspects of light.

Young correctly proposed that light was a transverse wave, while at the time some physicists erroneously thought that light was longitudinal (like a slinky). As a physician with interests in perception, Young proposed a model for color vision – a dichromatic theory with three cones.

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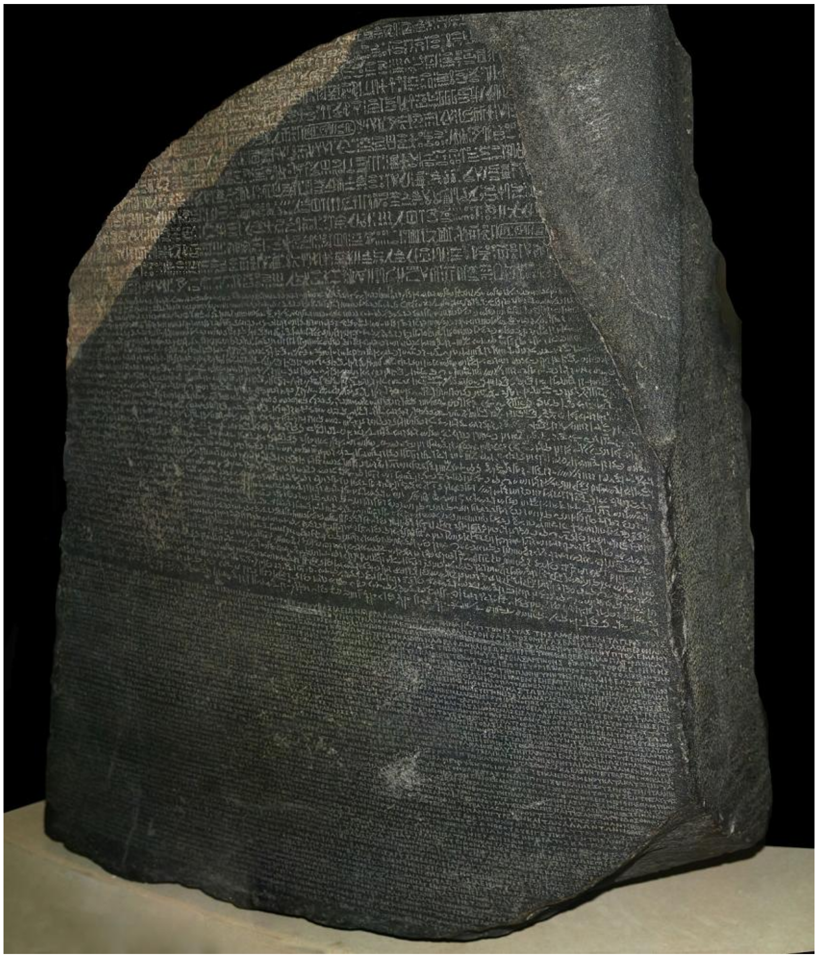
Thomas Young and the Rosetta Stone



The Rosetta Stone in the British Museum, London

© Hans Hillewaert

The ***Rosetta Stone***, a black tablet with engravings dating back to around 200 BC, was discovered near Rashid (Rosetta), Egypt, by Napoleon's army in 1799. It had a message repeated in three languages. The understanding of the hieroglyphic portion presented the greatest challenge.

Young became interested in this basalt rock in 1814 and after a few years had virtually unlocked the key, making a significant contribution to our understanding of this ancient Egyptian sacred writing. No wonder Young was called *Phenomena Young* by his classmates during his earlier university days at ***Cambridge University***.

Fresnel and The Competition

A competition was held by the French Academy of Sciences in 1818 calling for papers to explain the nature of light. The judges were mostly fans of the particle model of light. They were called corpuscularists – Laplace, Biot, and Poisson. The five judges were

Pierre-Simon Laplace (1749 - 1827) – French polymath;
Jean-Baptiste Biot (1774 - 1862) – French physicist, mathematician, astronomer;
Baron Siméon Denis Poisson (1781 – 1840) – French mathematician, engineer, physicist;
Joseph Louis Gay-Lussac (1778 – 1850) – French chemist and physicist;
Dominique-François-Jean Arago (1786 – 1853) – French mathematician, physicist, astronomer.

Fresnel entered with a paper on the wave nature of light. Poisson, among the corpuscularists was not impressed. He used Fresnel's wave theory to predict that a spot of light would be seen behind an obstacle and believed he had found the silver bullet to sink Fresnel's theory. How can a spot of light exist behind an object and even in the center of the shadow at that!

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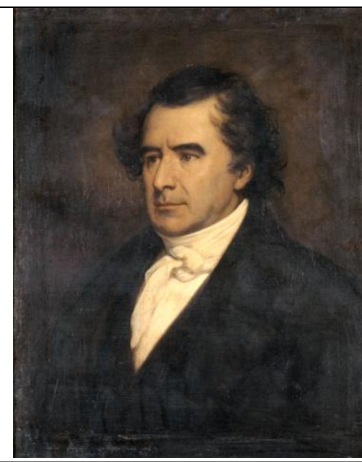
But the judge Arago was not so quick to judgment. He decided to set up an experiment to see if he could find a spot behind a small obstacle that blocked a source of light. And indeed he found it. Fresnel was redeemed and he won the prize!

They named the spot after Poisson, who theoretically used Fresnel's wave theory to predict the spot, and after Arago, who did the experiment to find the spot. I do not think Poisson was pleased. Poisson is know today among his discoveries: his Poisson distribution in statistics, the Poisson equation in physics, and the infamous Poisson spot!

The Poisson spot is also called the Arago spot in honor of Arago, who must have been very pleased to find it experimentally. The spot is also referred to as the Fresnel spot.

The competition event resulted in a dramatic success for the wave theory of light. You might say that the modern age of physical optics had begun.

By the way, Arago, later was the Prime Minister of France briefly in 1848. That year was unstable in Europe, as well as 1830, for revolutions abounded as people wanted their rights.

		
Augustin-Jean Fresnel 1788 - 1827	Siméon Denis Poisson 1781 - 1840	François Arago 1786 - 1853
French Civil Engineer, Physicist, Physical Optics	French Mathematician, Physicist, Engineer, Distribution in Statistics	French Mathematician, Physicist, Astronomer, French Prime Minister (1848)
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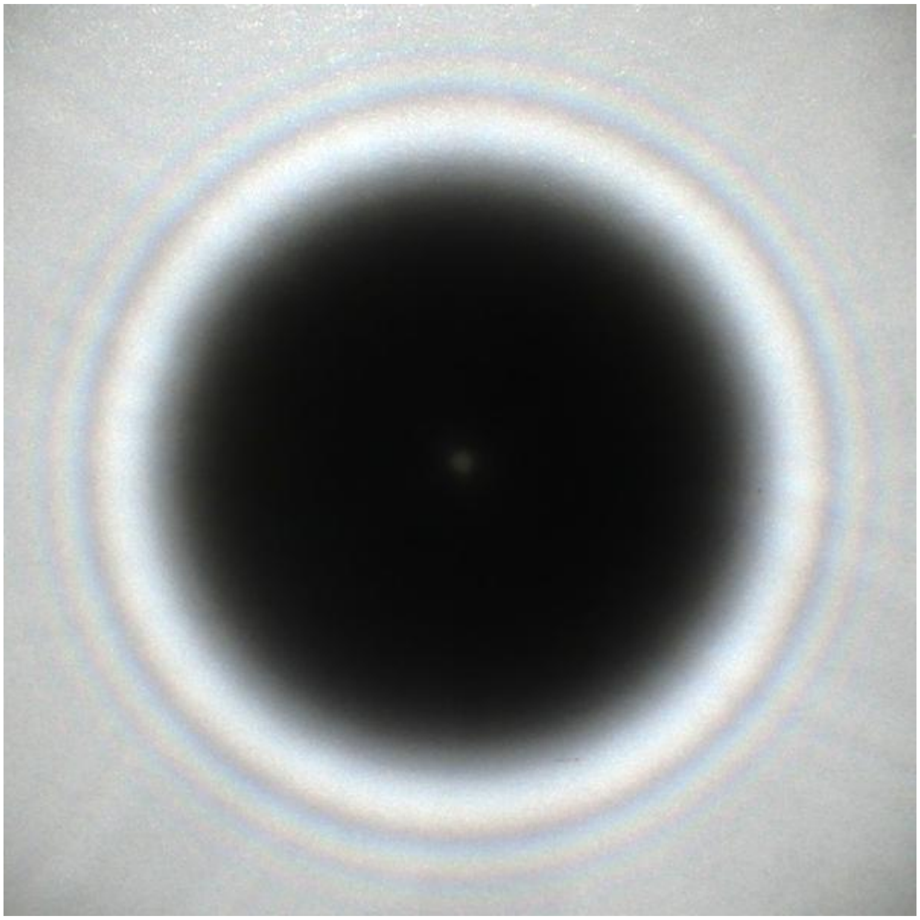
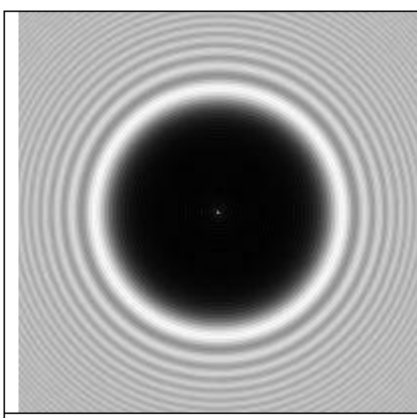
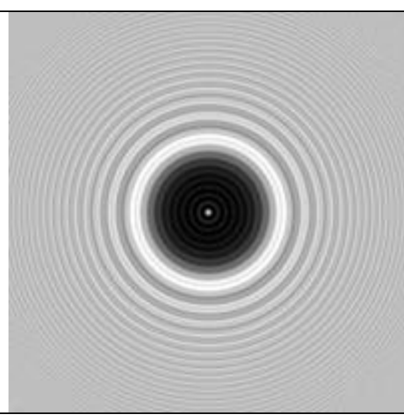
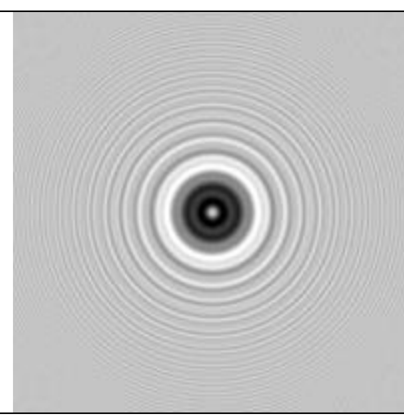
Poisson Spot, Arago Spot, Fresnel Spot in Shadow of 5.8 mm Disk



Wikipedia: Aleksandr Berdnikov. Creative Commons

Simulation with 633 nm He-Ne Laser Point Source 1 m Behind the Disk

		
Disk Diameter = 4 mm	Disk Diameter = 2 mm	Disk Diameter = 1 mm

Wikipedia: Thomas Reisinger. Creative Commons

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