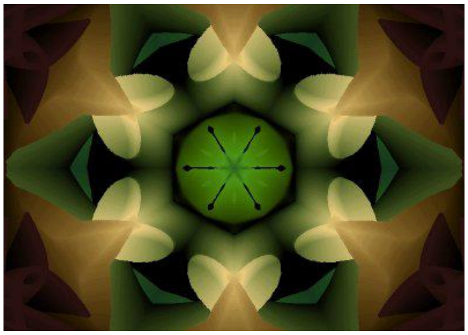
Modern Optics, Prof. Ruiz, UNCA **Fall 2020**

Exam 3 Closed Book, Closed Notes, Closed Everything, HONOR CODE
NOTE EXCEPTION: For this specific exam, a calculator is permitted.

Place a Box around all Final Answers to Each Part of the Subjective Questions.
For Full Credit in the Subjective Questions Show All Work.

SOLUTIONS

[30 pts] Multiple Choice GRE Style.



[30 pts] Multiple Choice GRE Style.

MC1. What is the angle between the two mirrors for this kaleidoscope? If you believe there is more than one angle that could be the answer, select the one with the smallest angle.

(A) 30° (B) 36° (C) 45° (D) 60° (E) 90°

There are 12 things that look like gum drops in the figure. Therefore 360°/12 = 30°. If you counted the 6 lines in the center circle or looked at the 6 outer green triangles you would be led to the wrong answer of 360°/6 = 60°. The error here is that you are counting pairs. It is best to go with the gumdrops since they are a little off the symmetry axes and that way you do not miscount a pair as one. The problem has a hint of go with the smallest angle to help prevent your counting a pair as one.

MC2. You have access to three types of polarizes:

V – a polarizer aligned for transmission of vertically polarized light
H – a polarizer aligned for transmission of horizontally polarized light
S – slanted polarizer aligned such that its polarization is 45° with respect to either of the other polarizers V and H.

The incident light from the Sun is unpolarized. You can say it has mixed polarization, i.e., polarizations in all possible directions superimposed. The light enters three polarizers that are placed one behind the other. Below are 5 cases of such filter arrangements where in two of the cases you use two of one type of the polarizers. For which case below will some light manage to get through all three filters?

(A) VVH (B) VHV (C) VHS **(D) VSH** (E) SVH

A V-polarizer placed right before an H or right after an H such as VH or HV kills off all the light. So you need the one that slips an S in between. Malus's Law $I(\theta) = I_o \cos^2\theta$ **is not really needed here. But you could use it. Note that** I_o **is the irradiance of a polarized beam of light coming towards the polarizer for which you want to use the formula. If you**

wanted to use Malus's Law, the first polarizer cuts light to 1/2 the original value and this light is now polarized. Call this polarized light I_o.

(A) VVH would be $I(\theta) = I_o I(0°) I(90°) = I_o \cos^2 0° \cos^2 90° = I_o \cdot 1 \cdot 0 = 0$

(B) VHV would be $I(\theta) = I_o I(90°) I(90°) = I_o \cos^2 0° \cos^2 0° = I_o \cdot 0 \cdot 0 = 0$

(C) VHS would be $I(\theta) = I_o I(90°) I(45°) = I_o \cos^2 0° \cos^2 45° = I_o \cdot 0 \cdot \frac{1}{2} = 0$

(D) VSH would be $I(\theta) = I_o I(45°) I(45°) = I_o \cos^2 45° \cos^2 45° = I_o \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{I_o}{4}$

(E) SVH would be $I(\theta) = I_o I(45°) I(90°) = I_o \cos^2 45° \cos^2 90° = I_o \cdot \frac{1}{2} \cdot 0 = 0$

MC3. Unpolarized light with an intensity of 1000 units enters a polarizer. It then passes through a second polarizer that is rotated such that its polarization axis is 45° with respect to that of the first. The intensity of the light after passing through the two polarizers is

(A) 0 units **(B) 250 units** (C) 500 units (D) 750 units (E) 1000 units.

When unpolarized light goes through the first polarizer, half the light gets through and this transmitted light is polarized. So light going through two filters can't be 1000 units or 750 units because these amounts are greater than 500 units. At 45° for the second filter, some light will get through the second filter. Therefore, the answer cannot be 0 units or 500 units. That leaves 250 units. You could use Malus's Law, but it is not needed. If you wanted to, you would have a $\cos^2 45° = \frac{1}{2}$ **factor working on the 50% that passed through the first filter, giving a final value of 1/4 of the original 1000 units for a final answer of 250 units.**

MC4. Four classical scenarios of a free electron in a laboratory setting are described below.

I. An electron is at rest.
II. An electron is traveling along the x-axis at constant speed.
III. An electron is accelerating along the x-axis.
IV. An electron is traveling along a circular path, i.e., along a circumference.

For which of the following cases does the electron emit electromagnetic radiation.

(A) I and II only (B) II, III, and IV only (C) II and III only
(D) III only **(E) III and IV only**

You need an accelerating charge to emit electromagnetic waves. When a mass travels in a circle of radius r with speed v, it has acceleration $a = \frac{v^2}{r}$**. So we need to include IV.**

MC5. Which vector below has zero divergence for all x and y?

(A) $x\hat{i} + y\hat{j}$ **(B)** $x\hat{i} - y\hat{j}$ (C) $x^2 y\hat{i}$ (D) $\cos x\hat{i} + \cos y\hat{j}$ (E) $e^x\hat{i} - e^y\hat{j}$

For (A) $\nabla \cdot (x\hat{i} + y\hat{j}) = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}) \cdot (x\hat{i} + y\hat{j}) = (\frac{\partial x}{\partial x}\hat{i} \cdot \hat{i} + \frac{\partial y}{\partial y}\hat{j} \cdot \hat{j}) = 1 \cdot 1 + 1 \cdot 1 = 2$

For (B) $\nabla \cdot (x\hat{i} - y\hat{j}) = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}) \cdot (x\hat{i} - y\hat{j}) = (\frac{\partial x}{\partial x}\hat{i} \cdot \hat{i} - \frac{\partial y}{\partial y}\hat{j} \cdot \hat{j}) = 1 \cdot 1 - 1 \cdot 1 = 0$

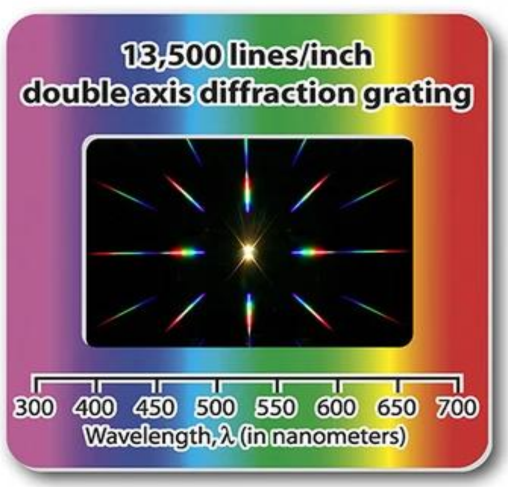
For (C) $\nabla \cdot (x^2 y\hat{i}) = (\hat{i}\frac{\partial}{\partial x}) \cdot (x^2 y\hat{i}) = \frac{\partial (x^2 y)}{\partial x}\hat{i} \cdot \hat{i} = y\frac{\partial (x^2)}{\partial x}\hat{i} \cdot \hat{i} = 2yx \cdot 1 = 2xy$

For (D) $\nabla \cdot (\cos x\hat{i} + \cos y\hat{j}) = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}) \cdot (\cos x\hat{i} + \cos y\hat{j}) = -\sin x - \sin y$

For (E) $\nabla \cdot (e^x\hat{i} - e^y\hat{j}) = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}) \cdot (e^x\hat{i} - e^y\hat{j}) = e^x - e^y$

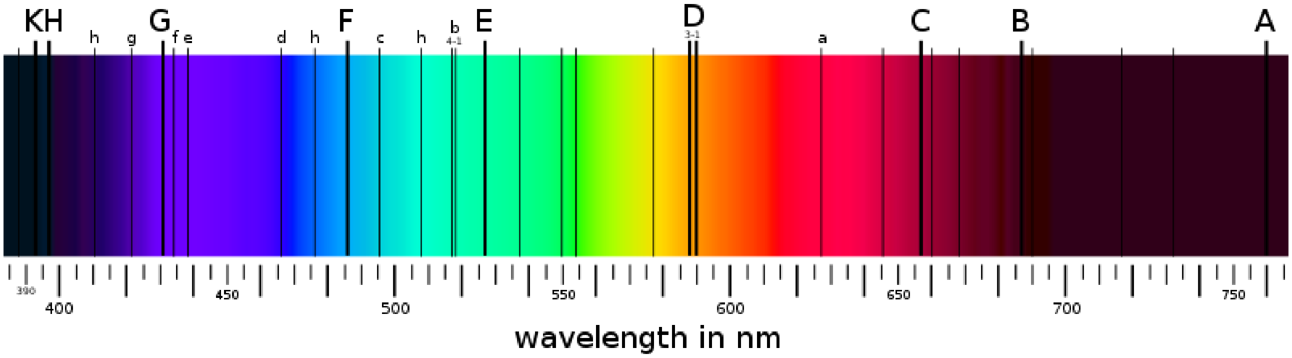
MC6. An observer holds up a diffraction grating to view a white light source across the room. The observer then looks at one of the first maxima which appear on either side of the central maximum, i.e., left or right. The color in the first maximum that is closest to the central beam is **(A)blue** (B)green (C) red.

When the wavelength gets smaller, the opening appears larger. Therefore, there is less diffraction.



Courtesy https://www.rainbowsymphonystore.com/

MC7. A spectrum can be observed using a spectrometer, where one type of design employs a diffraction grating. The ability to distinguish closely spaced lines is called the resolution.



Suppose that two nearby wavelengths λ_1 and $\lambda_2 > \lambda_1$ can just be distinguished, i.e. barely resolved. Define $\lambda_{avg} = \dfrac{\lambda_1 + \lambda_2}{2}$ and $\Delta\lambda = \lambda_2 - \lambda_1$. Which of the following could be used as a measure of resolving ability or power? A high resolving power means you can distinguish close wavelengths better.

(A) $\lambda_{avg} + \Delta\lambda$ (B) $\lambda_{avg} - \Delta\lambda$ **(C)** $\lambda_{avg}\Delta\lambda$ **(D)** $\dfrac{\lambda_{avg}}{\Delta\lambda}$ (E) $\dfrac{\Delta\lambda}{\lambda_{avg}}$

You want a small $\Delta\lambda$ to have a good rating. So $\dfrac{\lambda_{avg}}{\Delta\lambda}$ is the only choice that gives a high rating for small $\Delta\lambda$. Two other nice features are 1) the resolution measure is better when the small $\Delta\lambda$ is small relative to large wavelengths and 2) the measure is dimensionless.

MC8. For $\alpha = \dfrac{5\pi}{2N}$ where N is very large, the approximation for $\left[\dfrac{\sin(N\alpha)}{N\sin\alpha}\right]^2$ is

(A) $\left[\dfrac{2}{5\pi}\right]^2$ (B) $\left[\dfrac{2N}{5\pi}\right]^2$ (C) $\left[\dfrac{5\pi}{2}\right]^2$ (D) $\left[\dfrac{5\pi}{2N}\right]^2$ (E) $\left[\dfrac{10}{\pi}\right]^2$

$$\left[\frac{\sin(N\alpha)}{N\sin\alpha}\right]^2 = \left[\frac{\sin(N\frac{5\pi}{2N})}{N\sin(\frac{5\pi}{2N})}\right]^2 = \left[\frac{\sin(\frac{5\pi}{2})}{N\sin(\frac{5\pi}{2N})}\right]$$

Now note that $\sin(\dfrac{5\pi}{2}) = 1$ and $\sin(\dfrac{5\pi}{2N}) \approx \dfrac{5\pi}{2N}$ for large N.

$$\left[\frac{\sin(N\alpha)}{N\sin\alpha}\right]^2 = \left[\frac{\sin(\frac{5\pi}{2})}{N\sin(\frac{5\pi}{2N})}\right]^2 \approx \frac{1}{\left[N(\frac{5\pi}{2N})\right]^2} = \frac{1}{\left[\frac{5\pi}{2}\right]^2} = \left[\frac{2}{5\pi}\right]^2$$

MC9. A wave has an amplitude $E_p = E_o(1+i)$. Which answer below gives the irradiance?

(A) $\frac{E_o^2}{4}$ (B) $\frac{E_o^2}{2}$ **(C)** E_o^2 (D) $2E_o^2$ (E) $4E_o^2$

The irradiance is given by the formula $I_p = \frac{1}{2}|E_p|^2$ when you have an amplitude E_p.

$$I_p = \frac{1}{2}|E_p|^2 = \frac{1}{2}E_o^2(1+i)(1-i) = \frac{1}{2}E_o^2(1+1) = E_o^2$$

MC10. The irradiance due to blocking all Fresnel zones of a wavefront except the first zone is found to be $I_1 > 0$ at some point on the central axis of the wavefront. Now you block all zones except the first two. The irradiance will **(A) decrease** (B) remain the same (C) increase.

The even zones work against the odd zones due to the phase differences.

[20 pts] P1. Fresnel Equations. In class we derived the following.

$$r_p = \frac{n_2\cos\theta_1 - n_1\cos\theta_2}{n_1\cos\theta_2 + n_2\cos\theta_1} \qquad r_s = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2}$$

For this problem light is going from air ($n_1 = 1$) into a medium with index of refraction $n_2 = n$.

(a) Calculate $R_s = r_s^2$ in terms of n for $R_p = r_p^2 = 0$.

(b) What is R_s as an exact fraction in its most reduced form if n is exactly $\frac{3}{2}$?

(c) Give your answer to (b) to three significant figures.

SOLUTION

$$r_p = \frac{n_2\cos\theta_1 - n_1\cos\theta_2}{n_1\cos\theta_2 + n_2\cos\theta_1} \rightarrow \frac{n\cos\theta_1 - \cos\theta_2}{\cos\theta_2 + n\cos\theta_1} = 0$$

=> $n\cos\theta_1 - \cos\theta_2 = 0$ => $\cos\theta_2 = n\cos\theta_1$

$$r_s = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \rightarrow \frac{\cos\theta_1 - n\cos\theta_2}{\cos\theta_1 + n\cos\theta_2}$$

$$r_s = \frac{\cos\theta_1 - n(n\cos\theta_1)}{\cos\theta_1 + n(n\cos\theta_1)} = \frac{1-n^2}{1+n^2}$$

$$R_s = r_s^2 = \left[\frac{1-n^2}{1+n^2}\right]^2$$ or the equivalent form below.

$$\boxed{R_s = \left[\frac{n^2-1}{n^2+1}\right]^2}$$

(b) $$R_s = \left[\frac{(3/2)^2-1}{(3/2)^2+1}\right]^2 = \left[\frac{(9/4)-1}{(9/4)+1}\right]^2 = \left[\frac{5/4}{13/4}\right]^2$$

$$R_s = \left[\frac{5}{13}\right]^2$$

$$\boxed{R_s = \frac{25}{169}}$$

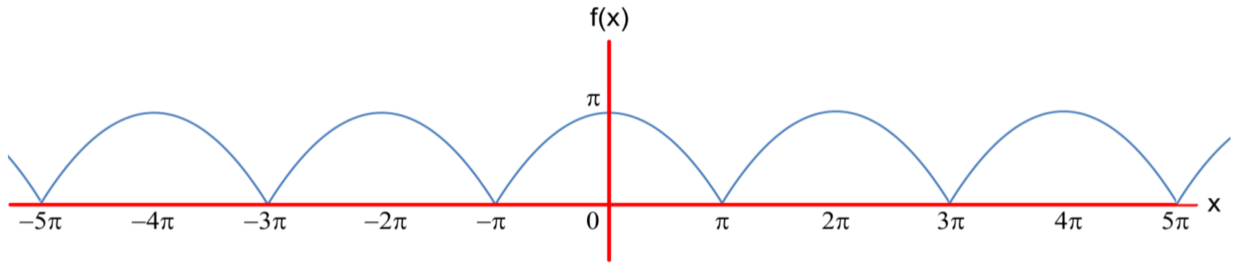
(c) $\boxed{R_s = 0.148}$

[20 pts] P2. Fourier Series. The basic formulas for a Fourier Series are given below.

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left[a_m \cos(mx) + b_m \sin(mx)\right]$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)\,dx$$



The periodic wave shown above is a repeating section of an inverted parabola.

(a) Use the general formula for a parabola $f(x) = ax^2 + bx + c$ to determine a , b , and c to fit the shown function in the interval from $x = -\pi$ to $x = +\pi$, the needed interval for the Fourier integrations. Write out $f(x)$ with your values for a, b, and c.

(b) Which of the Fourier coefficients are zero?

(c) Calculate a_0 only and you are finished with this problem. Give your answer in the simplest exact form.

SOLUTION

(a) $$f(x) = ax^2 + bx + c \quad \Rightarrow \quad f(0) = \pi \quad \Rightarrow \quad c = \pi$$

$$f(-x) = f(+x) \quad \Rightarrow \quad b = 0$$

$$f(\pm\pi) = 0 \quad \Rightarrow \quad a\pi^2 + \pi = 0 \quad \Rightarrow \quad a = -\frac{1}{\pi}$$

$$\boxed{f(x) = -\frac{1}{\pi}x^2 + \pi}$$

(b) The coefficients b_n are zero since the periodic wave is an even function and the b_n coefficients go with the old functions $\sin(nx)$. **EXTRA CREDIT WAS GIVEN IF YOU BURNED UP TIME CHECKING FOR ANY OF THE a_n BEING ZERO.**

(c) $$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{+\pi}\left[-\frac{x^2}{\pi} + \pi\right]dx$$

$$a_0 = \frac{1}{\pi}2\int_{0}^{+\pi}\left[-\frac{x^2}{\pi} + \pi\right]dx$$

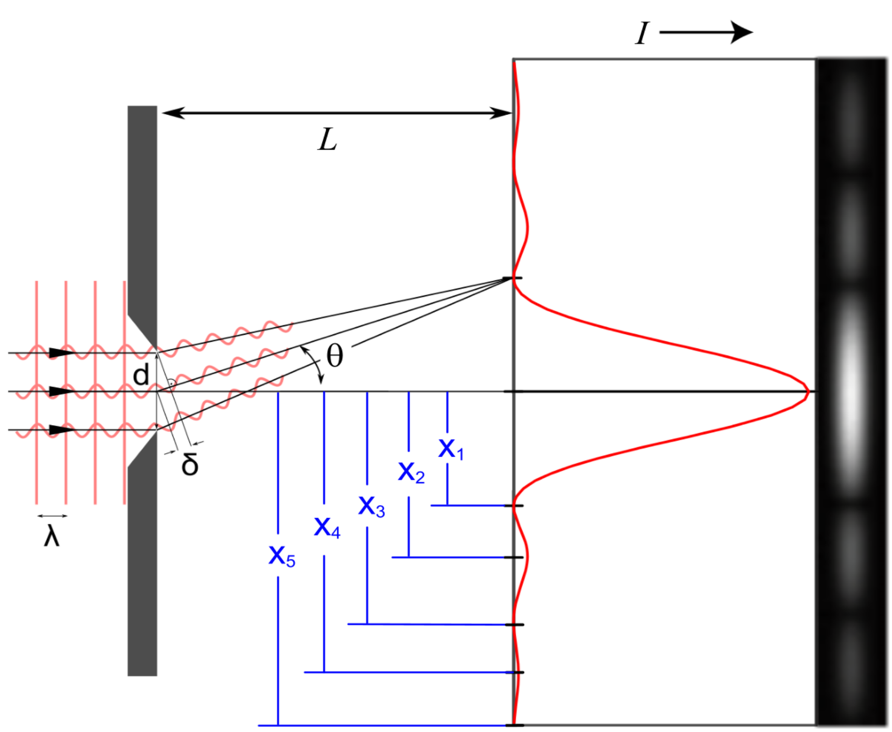
$$a_0 = \frac{2}{\pi}\left[-\frac{x^3}{3\pi} + \pi x\right]_0^{\pi} = \frac{2}{\pi}\left[-\frac{\pi^3}{3\pi} + \pi\pi\right] = 2\pi\left[-\frac{1}{3}+1\right] = 2\pi\,\frac{2}{3}$$

$$\boxed{a_0 = \frac{4\pi}{3}}$$

[20 pts] P3. Diffraction. Monochromatic light with wavelength λ goes through a slit with width $d = 160\lambda$ and reaches a screen a distance $L = 1\ \mathrm{m}$ after passing through the slit. The single-slit diffraction formulas are

$$I_r \equiv \frac{I(\theta)}{I_o} = \frac{\sin^2\beta}{\beta^2} \qquad \text{and} \qquad \beta = \frac{1}{2}kd\sin\theta\ ,$$

where θ is the usual angle measured from the center of the slit referenced to the axis joining the slit center and screen center (see figure). The two maxima x_2 and x_4 are not quite in the exact middle between their neighboring minima, but you may take the two maxima to be exactly in the middle of their neighboring minima to get fast results. Therefore, for example, you can use the very good approximations that $x_2 = \frac{1}{2}(x_1 + x_3)$ and $\beta_2 = \frac{1}{2}(\beta_1 + \beta_3)$.



Adapted from Wikipedia: jkrieger.

Complete the table below to two significant figures. You only need to do x_1 and x_2.
But note that you will need β_3 to complete the entry for 2.

	1	2	3	4	5
***x* (in mm)**			not applicable	not applicable	not applicable
I/I_o			not applicable	not applicable	not applicable

SOLUTION

We will do the minima case first, i.e., x_1. For this minimum we know that $I_r = 0$ and can fill in one of the boxes.

	1	2
***x* (in mm)**		
I_r	0.0	

To find x, we note that $\sin\theta = \dfrac{x}{L}$.

We will need the angle for the first minimum. So we want $\beta_1 = \pi$ in our equations

$$I_r = \frac{\sin^2\beta}{\beta^2} \quad \text{and} \quad \beta = \frac{1}{2}kd\sin\theta .$$

Since we are given the wavelength and d in terms of the wavelength, it is convenient to get β in the form with the wavelength explicitly shown,

$$\beta = \frac{1}{2}kd\sin\theta = \frac{1}{2}\frac{2\pi}{\lambda}d\sin\theta = \frac{\pi}{\lambda}d\sin\theta .$$

Furthermore, the given $d = 160\lambda$ leads to $\beta = \dfrac{\pi}{\lambda}160\lambda\sin\theta = 160\pi\sin\theta$.

This equation will be useful more than once: $\boxed{\beta = 160\pi\sin\theta}$.

Since $\beta_1 = \pi$, then $\pi = 160\pi\sin\theta_1$ => $\sin\theta_1 = \dfrac{1}{160}$.

$$x_1 = L\sin\theta_1 = 1\text{ m}\cdot\frac{1}{160} = 0.0063\text{ m} = 6.3\text{ mm}$$

$$\boxed{x_1 = 6.3\text{ mm}}$$

	1	2
***x* (in mm)**	6.3	
$\boldsymbol{I_r}$	0.0	

By the approximation given in the problem, the first maximum can be taken to be exactly between the first two minima.

$$\beta_2 = \frac{1}{2}(\beta_1 + \beta_3) = \frac{1}{2}(\pi + 2\pi) = \frac{3\pi}{2}$$

Using the formula $\beta = 160\pi \sin\theta$, we obtain

$$\beta_2 = \frac{3\pi}{2} = 160\pi \sin\theta_2 \, .$$

Therefore, $\frac{3}{2} = 160 \sin\theta_2$.

$$\sin\theta_2 = \frac{3}{320}$$

$$x_2 = L\sin\theta_2 = 1\text{ m} \cdot \frac{3}{320} = 0.0094\text{ m} = 9.4\text{ mm}$$

	1	2
***x* (in mm)**	6.3	9.4
$\boldsymbol{I_r}$	0.0	

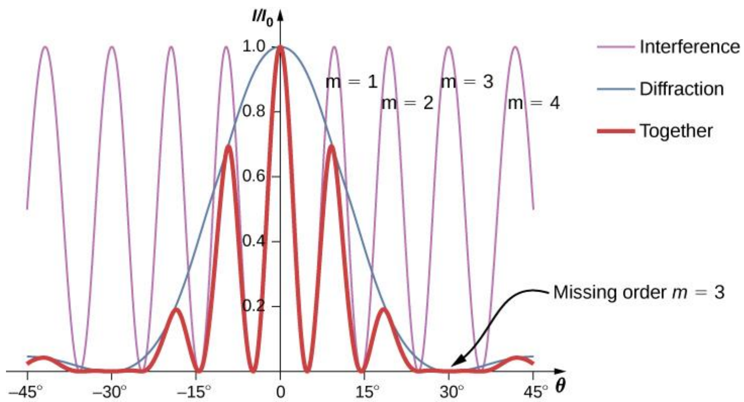
For the last entry we want the irradiance for $\beta_2 = \frac{3\pi}{2}$.

$$I_r = \left[\frac{\sin\beta}{\beta}\right]^2 = \left[\frac{\sin(3\pi/2)}{3\pi/2}\right]^2 = \left[\frac{-1}{3\pi/2}\right]^2 = \left[\frac{2}{3\pi}\right]^2 = 0.0450$$

	1	2
***x* (in mm)**	6.3	9.4
$\boldsymbol{I_r}$	0.0	0.045

[10 pts] P4. Interference. Light with wavelength λ enters two closely spaced slits where the distance between the centers of each slit is $a = 6\lambda$ and the width of each slit is $b = 2\lambda$. The relevant equations for the irradiance and associated parameters are

$$I_r \equiv \frac{I(\theta)}{I_o} = \cos^2\alpha \frac{\sin^2\beta}{\beta^2}, \quad \alpha = \frac{1}{2}ka\sin\theta, \text{ and } \beta = \frac{1}{2}kb\sin\theta.$$



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The first maximum to the right of the central maximum is slightly to the left of location of the first maximum m = 1 due to the interference factor. You may neglect this slight difference and consider the "together" max to be exactly at m = 1. Find the relative irradiance I_r for this "together" peak at the m = 1 interference location. Give your answer to three significant figures.

SOLUTION. The first maximum to the right of the central maximum due to the interference pattern occurs when we get $\cos^2\alpha = 1$ due to $\alpha = \pi$.

This result leads to $\alpha = \frac{1}{2}ka\sin\theta = \frac{1}{2}\frac{2\pi}{\lambda}a\sin\theta = \pi$.

$$\frac{1}{\lambda}a\sin\theta = 1 \quad => \quad \sin\theta = \frac{\lambda}{a} = \frac{\lambda}{6\lambda} = \frac{1}{6}$$

Now use $\beta = \frac{1}{2}kb\sin\theta$ and find $\beta = \frac{1}{2}kb\sin\theta = \frac{1}{2}\frac{2\pi}{\lambda}b\frac{1}{6} = \frac{1}{2}\frac{2\pi}{\lambda}2\lambda\frac{1}{6} = \frac{\pi}{3}$.

$$I_r = \cos^2\alpha\frac{\sin^2\beta}{\beta^2} = \cos^2\pi\frac{\sin^2(\pi/3)}{(\pi/3)^2} = \frac{9}{\pi^2}\left(\frac{\sqrt{3}}{2}\right)^2$$

$$I_r = \frac{27}{4\pi^2} \qquad \boxed{I_r = 0.684}$$