Theoretical Physics
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter A Notes. Taylor Series, Rotation Matrix, Groups

A1. Taylor Series

Brook Taylor (1685-1731)
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The Taylor series is a way to represent a function f(x) as a sum of powers of x. It is one of the basic tools of the physicist and is important in a study of any branch of physics. Here is the idea.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \ldots + a_n x^n + \ldots$$

We can shape the coefficients like a sculptor. For example,

$$f(0) = a_0 .$$

By taking derivatives and then setting x = 0 we can determine the coefficients a_n . Take the first derivative to arrive at

$$f'(x) = f^{(1)}(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \ldots + n a_n x^{n-1} + \ldots$$

$$f^{(1)}(0) = a_1$$

Taking the 2nd derivative gives

$$f''(x) = f^{(2)}(x) = 2a_2 + 3 \cdot 2 a_3 x + 4 \cdot 3 a_4 x^2 \ldots + n(n-1) a_n x^{n-2} + \ldots$$

$$f^{(2)}(0) = 2a_2$$

You can see the pattern here:

$$f^{(3)}(0) = 3 \cdot 2\, a_3 = 3 \cdot 2 \cdot 1 a_3 = 3! a_3$$

$$f^{(n)}(0) = n!a_n$$

Therefore, the “sculpted” coefficients are found from

$$a_n = \frac{f^{(n)}(0)}{n!}$$

We can then write

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$. or

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + ... + \frac{f^{(n)}(0)}{n!}x^n + ...$$

This is also called the Maclaurin Series. The Taylor Series has a more general form, where you can expand about any point x = a:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + ...$$

Note that the same trick works here as long as you evaluate the function at the point a.

PA1 (Practice Problem). Find the following Taylor series for the functions about x = 0.

It is good to memorize the results below from their patterns.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^4 + ...$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ...$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

Here is the solution for $\sin x$. We start with $f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

or

$$f(x) = f^{(0)}(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

So our function $f(x) = \sin x$. First find $f^{(0)}(0) = f(0) = \sin(0) = 0$. Then take the first derivative, $f^{(1)}(x) = \cos x$ to arrive at $f^{(1)}(0) = \cos(0) = 1$ and continue along these lines. See the table below.

$f^{(0)}(x) = \sin x$	$f^{(0)}(0) = \sin(0) = 0$	$f(0) = 0$
$f^{(1)}(x) = \cos x$	$f^{(1)}(0) = \cos(0) = 1$	$f^{(1)}(0)x = x$
$f^{(2)}(x) = -\sin x$	$f^{(2)}(0) = -\sin(0) = 0$	$f^{(2)}(0)\frac{x^2}{2!} = 0$
$f^{(3)}(x) = -\cos x$	$f^{(3)}(0) = -\cos(0) = -1$	$f^{(3)}(0)\frac{x^3}{3!} = -\frac{x^3}{3!}$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = \sin(0) = 0$	$f^{(4)}(0)\frac{x^4}{4!} = 0$
$f^{(5)}(x) = \cos x$	$f^{(5)}(0) = \cos(0) = 1$	$f^{(5)}(0)\frac{x^5}{5!} = \frac{x^5}{5!}$

Now add that last column so that

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$ becomes

$$f(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \ldots$$

Can you psyche out the next few terms by the patterns?

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots$$

PA2 (Practice Problem). Start with

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 ... + a_n(x-a)^n + ...$$

and show that

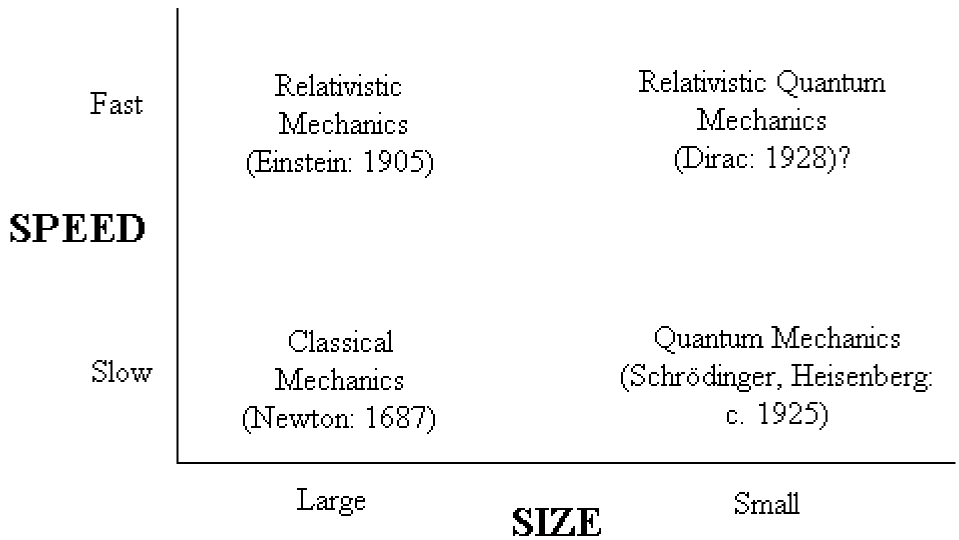
$$f(x) = f(a) + f\,'(a)(x-a) + \frac{f\,''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + ...$$

using steps similar to that which we did for the expansion about the origin.

A2. A Use of the Taylor Expansion

Here is a summary of basic laws of motion in physics. Note that we do not have a complete theory of relativistic quantum mechanics at this time. That is why we include the question mark.

Classical and Modern Physics



The Dirac equation is not usually touched on in undergraduate courses. Dirac's work inspired Richard Feynman (as well as Schwinger and Tomonaga) to develop quantum electrodynamics, an extension of Dirac's relativistic quantum mechanics for the electromagnetic interaction.

Brook Taylor again (1685-1731)
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Here is our British mathematician Taylor again, this time looking more formal in his fancy wig.

When a fine theory like Newtonian mechanics is established, yet found to need improvement, a new theory comes along to extend it. At the same time the new theory embraces the old, meaning, that in the proper limit, we can recover the old theory. This is an elegant example of the beauty of physics. In other words, you recover Newtonian mechanics from Einstein's relativistic mechanics when you slow your speeds down. Slowing down means getting at least slower than 1/10 the speed of light.

By the way, "theory" in physics is used like "theory" in "music theory," an established body of knowledge backed up by observation and analysis. However, a theory can have limitations, e.g., Newton's theory, "classical mechanics," applies to the world of the large and slow moving.

Define $\beta = \frac{v}{c}$. Imagine some calculated quantity $f(\beta)$ from relativity:

$$f(\beta) = f(0) + f^{(1)}(0)\beta + f^{(2)}(0)\frac{\beta^2}{2!} + f^{(3)}(0)\frac{\beta^3}{3!} + \ldots$$

For $\beta = 0.1$, the first relativistic correction is on the order of 1/10, the second 1/100, etc. The classical term is $f(0)$. Two examples of relativistic formulas are those for Lorentz contraction and time dilation, which appear below.

$$L = L_0\sqrt{1-\beta^2}$$

$$T = \frac{T_0}{\sqrt{1-\beta^2}}$$

These can be written in the following forms.

$$\frac{L}{L_0} = \sqrt{1-\beta^2} = (1-\beta^2)^{1/2} \quad \text{and} \quad \frac{T}{T_0} = \frac{1}{\sqrt{1-\beta^2}} = (1-\beta^2)^{-1/2}$$

PA3 (Practice Problem). Use your results from PA1 for expanding $(1+x)^n$, where $x = -\beta^2$ and $n = \frac{1}{2}$ or $n = -\frac{1}{2}$ to give the coefficients for the first few terms of each expansion below:

$$\frac{L}{L_0} = \sqrt{1-\beta^2} = (1-\beta^2)^{1/2} \quad \text{and} \quad \frac{T}{T_0} = \frac{1}{\sqrt{1-\beta^2}} = (1-\beta^2)^{-1/2}$$

A3. Matrices

Here is a review of the basic rules for matrix multiplication.

Multiplying Two Matrices

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Multiplying a Matrix and a Column Matrix

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

PA4 (Practice Problem). Consider the following three matrices called the Pauli matrices.

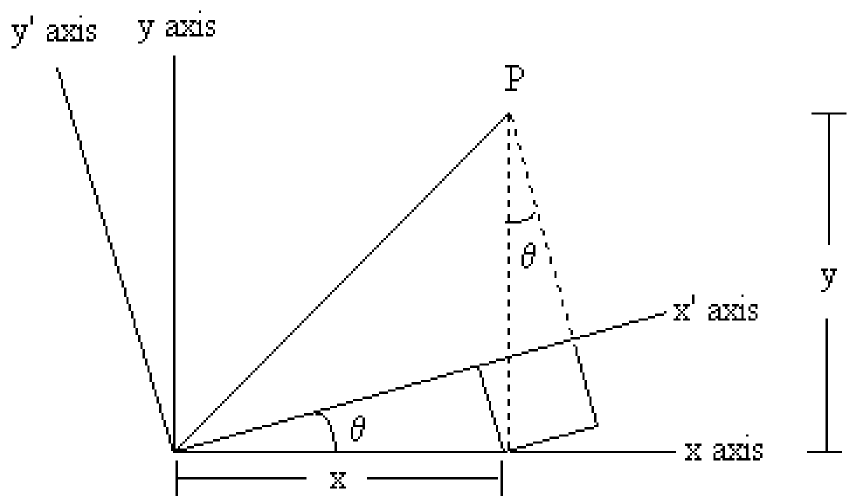
$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

What do you get for $\sigma_x \sigma_y$?

What about multiplying these in reverse order: $\sigma_y \sigma_x$?

Try the same for another pair.

A4. The Rotation Matrix



There are two axes above. The prime axes are rotated with respect to the original x-y system. The secret in relating (x', y') to (x, y) is to construct that cute rectangle you see in the above figure. Note that

$$x' = x\cos\theta + y\sin\theta \text{ and } y' = y\cos\theta - x\sin\theta.$$

Write these as

$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

We can define the following matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

In matrix notation we can write

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R(\theta)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

A5. Trig Identities

Now the fun begins and we see our first example of the power of theoretical physics. We will proceed to derive complicated trig formulas in one step.

Remember those days, perhaps in high school, when you first encountered complicated trig identities involving the sines and cosines of sums and differences of angles. Here you can derive these quickly. The combined rotation

$$R(\alpha+\beta) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

has to be equal to

$$R(\alpha+\beta) = \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}.$$

Multiply the matrices and your a_{11} matrix element is your cosine identity for $\cos(\alpha+\beta)$. The element a_{12} takes care of $\sin(\alpha+\beta)$. The results are:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta.$$

Replace β with $-\beta$ and you arrive at the formulas involving the differences. Remember that the cosine is an even function, i.e., $\cos(-\beta) = \cos\beta$, and the sine is an odd function such that: $\sin(-\beta) = -\sin\beta$.

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

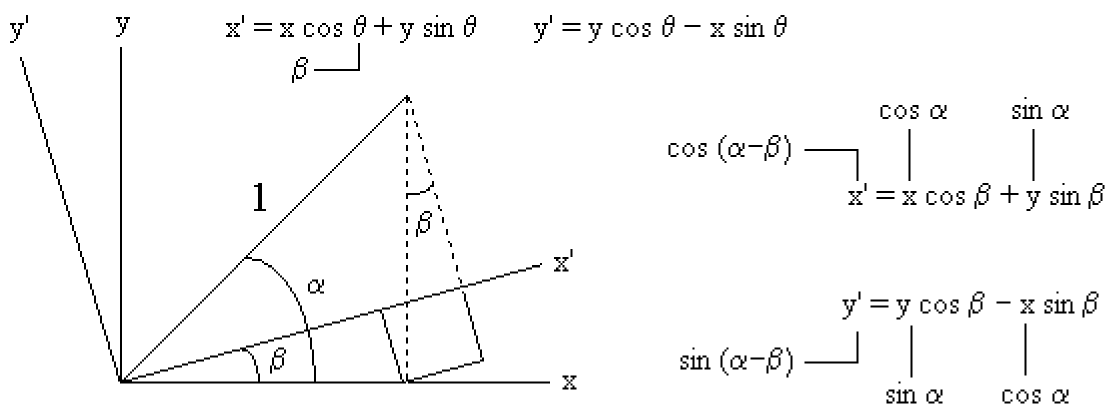
$$\sin(\alpha-\beta) = -\cos\alpha\sin\beta + \sin\alpha\cos\beta.$$

This is an example of the "magic" of theoretical physics. You might say these are Feynmanesque derivations. We get the result in a couple of lines, while the high school proof goes on and on with intricate diagrams and multiple algebraic steps that can take over a page.

In fact, we are not even afraid of the triple-angle sum, $\cos(\alpha+\beta+\gamma)$. Just multiply another matrix and pick off the appropriate part. This is another characteristic of Feynman - using theoretical techniques to do even more general and more difficult proofs with relative ease.

A6. Visualization of a Trig Identity

For the angle-difference case, our rotation diagram provides us with a nice picture to visualize the pieces found in the identities.



$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

PA5 (Practice Problem). Use the above formulas to derive the result for $\tan(\alpha-\beta)$. Then replace β with $-\beta$ to arrive at the identity for $\tan(\alpha+\beta)$. Arrange your results to look like the standard forms:

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

and

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}.$$

A7. Groups

Group: A set of elements $G = \{a, b, c, d ...\}$ with a binary operation "$\cdot$", i.e., represented by a dot satisfying the following conditions. The symbol $a \in G$ means that a is an element in G.

1. Closure: for $a \in G$ and $b \in G$, $a \cdot b \in G$.

2. Association: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

3. Identity Element: $I \in G$ such that $a \cdot I = a$.

4. Inverse Element: $a^{-1} \in G$ such that $a \cdot a^{-1} = I$.

The group is an Abelian group if commutation holds: $a \cdot b = b \cdot a$.

Note that we can prove $I \cdot a = a$ from $a \cdot I = a$ and the other group properties. In words, we are saying the we can prove that our right identity element is also a left identity element from the group properties. When one says an element is an identity element, this means both left and right identity, i.e., using it on the left side or right side.

We will work backwards starting with the result and transforming it into something we know is true. Then you can copy the steps backwards as the proof.

$$I \cdot a = a$$

$$I \cdot a \cdot a^{-1} = a \cdot a^{-1}$$

$$I \cdot (a \cdot a^{-1}) = I$$

$$I \cdot I = I$$

This last equation is true because you can consider the Identity working on the right side.

PA6 (Practice Problem). In a similar way prove that the inverse of "a" works as an inverse on the left side too, i.e., the inverse is a right and left inverse. It doesn't matter on which side it appears in the binary operation. In summary, prove $a^{-1} \cdot a = I$.

A8. The Military Group and the Square Root of Minus One

PA7 (Practice Problem). Below is a delightful example from a magazine called *Quantum*: Alexey Sosinsky, "Marching Orders", *Quantum* Vol. 2, No. 2, 7 (1991).

Consider a finite group of 4 elements: G = {A, B, L, R}

A: Attention (maintain the direction you are facing)
B: 'Bout Face (turn around)
L: Left Face (turn to face left)
R: Right Face (turn to face right)

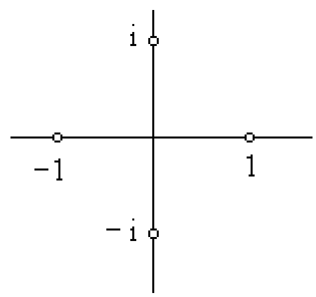
We will call this the “military group.” Consider doing an ‘Bout face and then standing at Attention. We write this as AB, placing the first instruction at the far right. The net result is AB = B, i.e., ‘Bout face. Complete the multiplication table shown here.

	A	B	L	R
A	A	B		
B			R	
L				A
R		L		

Are all the conditions for a group met? Is the group Abelian?

PA8 (Practice Problem). Have you ever pondered about the meaning of $i = \sqrt{-1}$ long ago? Well, you now have profound understanding since our “military group” has an *isomorphism*, i.e., one-to-one correspondence, to another small group involving imaginary numbers. Check out the group

$G = \{1, -1, i, -i\}$ under the binary operation of multiplication.



Construct the multiplication table for its elements. Compare with the “military group,” identifying the isomorphism for the elements A, B, L, and R. So, what does it mean to multiply by 1 by $i = \sqrt{-1}$?

Label each of these A, B, R, and L to explicitly describe the isomorphism.

Since you have complete understanding of the "Military Group" and this group is isomorphic to $G = \{1, -1, i, -i\}$ under the binary operation of multiplication, you have complete understanding of $i = \sqrt{-1}$. It is a rotation! You are led by the isomorphism to define a complex plane and place $i = \sqrt{-1}$ after making a 90° turn (i.e., left turn) from 1 on the real axis.

PA9 (Practice Problem). Evaluate i^{100} , i^{101} and $(-i)^{102}$. How does the isomorphism with the "Military Group" allow you to determine these quickly in your head?