# Theoretical Physics
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## Chapter K Homework. The Pauli Equation

**HW-K1. Matrix in an Exponential.** We define an matrix in an exponential as follows:

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots$$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and

$A^2 = AA$, $A^3 = AAA$, and so on (matrix multiplication).

Give the simplest result for $e^A$ for $A = \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

**Solution.** $$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots$$

We need to calculate the powers of the matrix $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$$M^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$$M^3 = -M \qquad M^4 = I \qquad M^5 = M$$

$$e^A = I + \theta M - \frac{\theta^2}{2!}I - \frac{\theta^3}{3!}M + \frac{\theta^4}{4!}I + \frac{\theta^5}{5!}M \ldots$$

$$e^A = \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right]I + \left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right]M$$

$$e^A = (\cos\theta)I + (\sin\theta)M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

**HW-K2. Wave Function with Spin.** Consider the wave function $\psi$ :

$$\psi = 0 \text{ for } -\infty \le x < 0 \qquad \psi = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}e^{-x/2} \text{ for } 0 \le x \le \infty .$$

and the Hermitian matrix operator: $A = \begin{bmatrix} x & i \\ -i & x \end{bmatrix}$. Calculate $\langle \psi | A | \psi \rangle$.

$$\langle \psi | A | \psi \rangle = \int_0^\infty \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix} e^{-x/2}\begin{bmatrix} x & i \\ -i & x \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}e^{-x/2}dx$$

$$\langle \psi | A | \psi \rangle = \frac{1}{2}\int_0^\infty \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix} x & i \\ -i & x \end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}e^{-x}dx$$

$$\langle \psi | A | \psi \rangle = \frac{1}{2}\int_0^\infty \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix} x + i \\ -i + x \end{bmatrix}e^{-x}dx$$

$$\langle \psi | A | \psi \rangle = \frac{1}{2}\int_0^\infty \left[x + i - i + x\right]e^{-x}dx$$

$$\langle \psi | A | \psi \rangle = \int_0^\infty x e^{-x}dx$$

$$\int_0^\infty e^{-\alpha x}dx = -\frac{1}{\alpha}e^{-\alpha x}\Big|_0^\infty = -\frac{1}{\alpha}\left[0-1\right] = \frac{1}{\alpha} \quad \text{where } \alpha > 0 .$$

$$\int_0^\infty x e^{-\alpha x}dx = -\frac{d}{d\alpha}\int_0^\infty e^{-\alpha x}dx = -\frac{d}{d\alpha}\frac{1}{\alpha} = \frac{1}{\alpha^2} \qquad \text{Let } \alpha = 1 .$$

$$\langle \psi | A | \psi \rangle = \int_0^\infty x e^{-x}dx = 1$$