Theoretical Physics
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Chapter P Homework. Fourier Transforms

P1-P3. Fourier Transforms. Calculate the Fourier transforms of the following three functions by **explicitly doing all integrations.** Give the simplest form for each answer.

HW-P1. $f(x) = e^{-ax}$ for $x \geq 0$ where $a > 0$ and $f(x) = 0$ for $x < 0$.

$$\Im\{f(x)\} \equiv F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-ax} e^{-ikx} dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-ax-ikx} dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-\alpha - ik} \right] e^{-ax-ikx} \Big|_{0}^{\infty}$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-\alpha - ik} \right] (0-1)$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha + ik} \right]$$ or, you can continue with

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha + ik} \right] \left[\frac{\alpha - ik}{\alpha - ik} \right] = \frac{1}{\sqrt{2\pi}} \frac{\alpha - ik}{\alpha^2 + k^2}$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \frac{\alpha - ik}{\alpha^2 + k^2} .$$

HW-P2. $f(x) = e^{-a|x|}$ for all x where $a > 0$.

$$\Im\{f(x)\} \equiv F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-a|x|} e^{-ikx} dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{+ax} e^{-ikx} dx + \int_{0}^{\infty} e^{-ax} e^{-ikx} dx \right]$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha - ik} \right] e^{ax-ikx} \Big|_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-\alpha - ik} \right] e^{-ax-ikx} \Big|_{0}^{\infty}$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha - ik} \right] (1-0) + \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-\alpha - ik} \right] (0-1)$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha - ik} \right] + \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha + ik} \right]$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha - ik} + \frac{1}{\alpha + ik} \right]$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{\alpha + ik + \alpha - ik}{(\alpha - ik)(\alpha + ik)} \right]$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \frac{2\alpha}{(\alpha^2 + k^2)}$$

HW-P3. $f(x)=\frac{1}{a}$ for $-\frac{a}{2}\leq x\leq\frac{a}{2}$ where $a>0$ and $f(x)=0$ elsewhere.

$$\Im\{f(x)\}\equiv F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(x)e^{-ikx}dx$$

$$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-a/2}^{+a/2}\frac{1}{a}e^{-ikx}dx$$

$$F(k)=\frac{1}{\sqrt{2\pi}}\frac{1}{a}\frac{e^{-ikx}}{(-ik)}\Bigg|_{-a/2}^{+a/2}$$

$$F(k)=\frac{1}{\sqrt{2\pi}}\frac{1}{a}\frac{\left[e^{-ika/2}-e^{ika/2}\right]}{(-ik)}$$

$$F(k)=\frac{1}{\sqrt{2\pi}}\frac{1}{a}\frac{\left[e^{ika/2}-e^{-ika/2}\right]}{ik}$$

$$F(k)=\frac{1}{\sqrt{2\pi}}\frac{2}{ka}\left[\frac{e^{ika/2}-e^{-ika/2}}{2i}\right]$$

$$F(k)=\frac{1}{\sqrt{2\pi}}\frac{2}{ka}\sin\left(\frac{ka}{2}\right)$$

$$F(k)=\frac{1}{\sqrt{2\pi}}\frac{\sin(ka/2)}{ka/2}=\frac{1}{\sqrt{2\pi}}\operatorname{sinc}\left(\frac{ka}{2}\right)$$

HW-P4. Heisenberg Uncertainty Relation. For the wave function $\psi(x) = (\alpha/\pi)^{\frac{1}{4}} e^{-\frac{\alpha}{2}x^2}$.

Calculate $\Delta x \Delta k$, i.e., $\sigma_x \sigma_k$. What happens to σ_x and σ_k if α increases?

$$\Im\{\psi(x)\} \equiv \chi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(x) e^{-ikx} dx$$

$$\chi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\alpha}{2}x^2} e^{-ikx} dx$$

Use $\int_{-\infty}^{\infty} e^{-Ax^2+Bx} dx = \sqrt{\frac{\pi}{A}} e^{\frac{B^2}{4A}}$, where $A = \frac{\alpha}{2} > 0$ and $B = -ik > 0$.

$$\chi(k) = \frac{1}{\sqrt{2\pi}} \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{\pi}{A}} e^{\frac{B^2}{4A}} = \frac{1}{\sqrt{2\pi}} \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{2\pi}{\alpha}} e^{-\frac{k^2}{2\alpha}} = \left(\frac{1}{\pi\alpha}\right)^{\frac{1}{4}} e^{-\frac{k^2}{2\alpha}}$$

$$P(x) = \psi^* \psi = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2} \qquad P(k) = \chi^* \chi = \sqrt{\frac{1}{\pi\alpha}} e^{-\frac{k^2}{\alpha}}$$

$$\sigma_x^2 = \int (x-\mu_x)^2 P(x)\, dx \qquad \sigma_k^2 = \int (k-\mu_k)^2 P(k)\, dk$$

Note that $\mu_x = 0$ and $\mu_k = 0$ since each distribution is centered on zero.

We will need $\int_{-\infty}^{\infty} z^2 e^{-Az^2} dz = \frac{1}{2}\sqrt{\frac{\pi}{A^3}}$, where $A > 0$.

$$\sigma_x^2 = \int_{-\infty}^{+\infty} x^2 P(x) dx = \int_{-\infty}^{+\infty} x^2 \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\alpha}{\pi}} \sqrt{\frac{\pi}{A^3}}\Bigg|_{A=a} = \frac{1}{2\alpha}$$

$$\sigma_k^2 = \int_{-\infty}^{+\infty} k^2 P(k) dk = \int_{-\infty}^{+\infty} k^2 \sqrt{\frac{1}{\pi\alpha}} e^{-\frac{k^2}{\alpha}} dk = \frac{1}{2}\sqrt{\frac{\alpha}{\pi}} \sqrt{\frac{\pi}{A^3}}\Bigg|_{A=1/a} = \frac{\alpha}{2}$$

$\sigma_x^2 \sigma_k^2 = 1/4$ and $\sigma_x \sigma_k = 1/2$ (minimum case for the uncertainty relation, the Gaussian)

If α increases, σ_x decreases, the x-position wave function is tall and thin (more certainty).

But σ_k then increases and the k-variable spread is short and wide (more uncertainty).

If you know the position with more certainty, there is less certainty in the momentum.