Theoretical Physics
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter S Notes. Cauchy Integral Formula

S1. Cauchy-Riemann Conditions. Below we list a constant complex number, a complex variable, and a complex function.

Complex Constant: $a+ib$, where $i=\sqrt{-1}$ with a and b real.

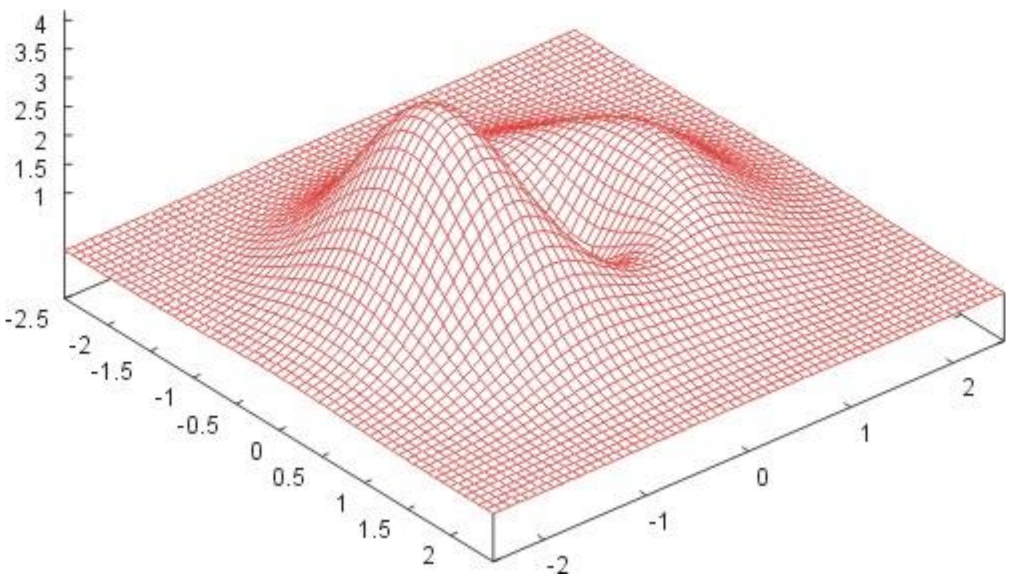
Complex Variable: $z=x+iy$, where $\{x, y \in R\}$

Complex Function: $f(z)=u(x, y)+iv(x, y)$, where $\{u, v \in R\}$

Question 1. Is differentiation well-defined for complex functions? Let's see.

$$f'(z)=\frac{df(z)}{dz}=\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$f'(z)=\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\left[u(x+\Delta x, y+\Delta y)-u(x, y)\right]+i\left[v(x+\Delta x, y+\Delta y)-v(x, y)\right]}{\Delta x+i\Delta y}$$



Courtesy Dr. David Volker Freie Universität Berlin

But we can approach our definition of slope from many directions. We want a unique derivative.

So we will evaluate the slope along the x-direction (East), then the y-direction (North).

Slope along the x-axis. Then $\Delta z = \Delta x$ with $\Delta y = 0$ and we obtain

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\left[u(x+\Delta x, y+\Delta y) - u(x,y)\right] + i\left[v(x+\Delta x, y+\Delta y) - v(x,y)\right]}{\Delta x + i\Delta y}$$

$$\left.\frac{df(z)}{dz}\right|_{\Delta y=0} = \lim_{\Delta x \to 0} \frac{\left[u(x+\Delta x, y) - u(x,y)\right] + i\left[v(x+\Delta x, y) - v(x,y)\right]}{\Delta x}$$

$$\left.\frac{df(z)}{dz}\right|_{\Delta y=0} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Slope along the y-axis. Then $\Delta z = i\Delta y$ with $\Delta x = 0$ and we obtain

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\left[u(x+\Delta x, y+\Delta y) - u(x,y)\right] + i\left[v(x+\Delta x, y+\Delta y) - v(x,y)\right]}{\Delta x + i\Delta y}$$

$$\left.\frac{df(z)}{dz}\right|_{\Delta x=0} = \lim_{\Delta y \to 0} \frac{\left[u(x, y+\Delta y) - u(x,y)\right] + i\left[v(x, y+\Delta y) - v(x,y)\right]}{i\Delta y}$$

$$\left.\frac{df(z)}{dz}\right|_{\Delta x=0} = \frac{\partial u}{i\partial y} + i\frac{\partial v}{i\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

For a unique derivative, we need

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}}$$

Augustin-Louis Cauchy (1789-1857) **Bernhard Riemann (1826-1866)**

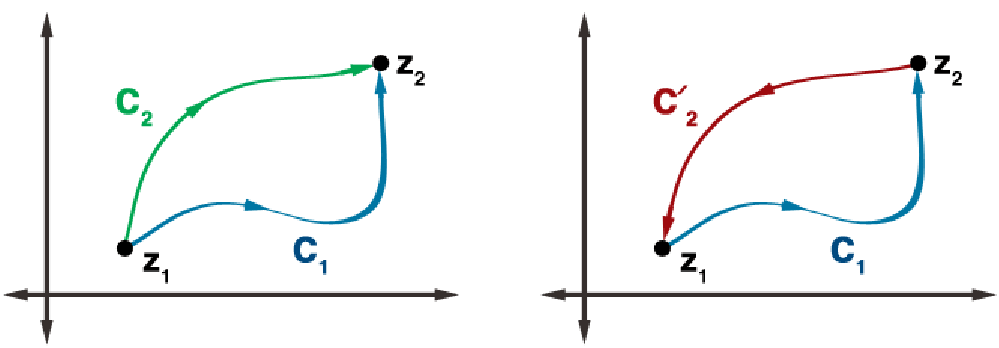
Courtesy School of Mathematics and Statistics, University of St. Andrews, Scotland

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

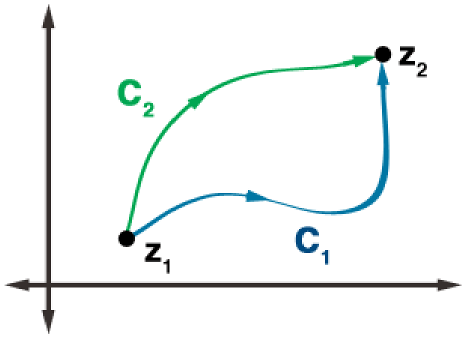
Question 2. Is integration path independent? Let's see. We consider this question in the next section.

S2. Green's Theorem.

For path independence, we consider two integral paths.



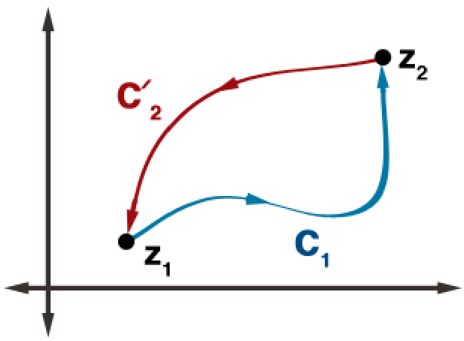
Contour Figures Courtesy Mathematical Methods in Physics, CUNY Graduate Center



Here is why this must be the case. For path independence

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$



Since

$$\int_{C_2'} f(z)\,dz = -\int_{C_2} f(z)\,dz\ ,$$

then

$$\int_{C_1} f(z)\,dz + \int_{C_2'} f(z)\,dz = 0\ .$$

In other words, integrating over a closed path gives zero.

$$\oint_C f(z)dz = 0$$

Our integral be written as

$$\oint_C f(z)dz = \oint_C (u+iv)(dx+idy)$$

$$\oint_C f(z)dz = \oint_C \left[(udx - vdy) + i(vdx + udy)\right]$$

We have two integrals of the type

$$I = \oint_C (Ldx + Mdy)$$

Consider a vector $\vec{B} = L\hat{i} + M\hat{j}$ and our usual radial vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Now take the dot product of $\vec{B} = L\hat{i} + M\hat{j}$ with $\vec{dr} = \hat{i}dx + \hat{j}dy + \hat{k}dz$ and integrate around a closed path. We obtain our integral of interest.

$$I = \oint_C (L\hat{i} + M\hat{j}) \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz) = \oint_C (Ldx + Mdy)$$

$$I = \oint_C \vec{B} \cdot \vec{dr}$$

Remember Stoke's Theorem?

$$\oint_C \vec{B} \cdot \vec{dl} = \iint_A (\nabla \times \vec{B}) \cdot \vec{dA}$$

Since our path is in the x-y plane, $d\vec{A} = \hat{k}dA = \hat{k}dxdy$.

$$\oint_C \vec{B} \cdot \vec{dl} = \iint_A \left[\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right] dxdy$$

The above is known as Green's Theorem. Think of it as a special case of Stoke's Theorem. In terms of our original integral

$I = \oint_C (Ldx + Mdy)$ with $\vec{B} = L\hat{i} + M\hat{j}$, we have the following.

$$\boxed{\oint_C (Ldx + Mdy) = \iint_A \left[\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right] dxdy}$$

The above is the usual form one finds for Green's Theorem in texts. For path independence we must have

$$\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} = 0 .$$

We now apply this to our integral in the complex plane.

$$\oint_C f(z)dz = \oint_C (u+iv)(dx+idy)$$

$$\oint_C f(z)dz = \oint_C \left[(udx - vdy) + i(vdx + udy)\right]$$

We want $\oint_C (Ldx + Mdy) = \iint_A \left[\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right] dxdy = 0$.

We apply this condition to the real part and then to the imaginary part, arriving at a pair of equations.

For the real part $L = u$ and $M = -v$. The condition $\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} = 0$ results in

$\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y} = 0$, which is the same condition as $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

For the imaginary part $L = v$ and $M = u$. The condition $\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} = 0$ results in

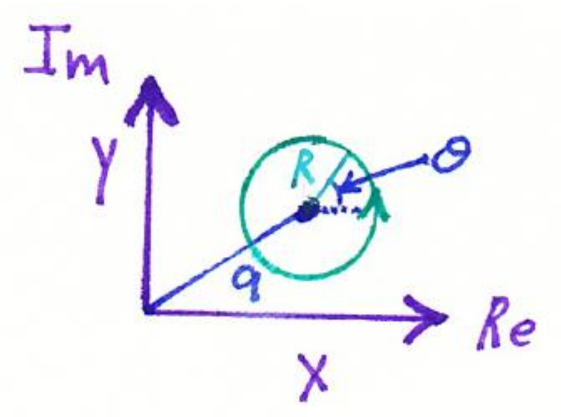
$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$, which is the same condition as $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$.

But these are the very same Cauchy-Riemann conditions that gave us a well-defined derivative. This is too good to be true. The same conditions make the derivative and also the integral work for us. When these conditions are met we say that the function $f(z)$ is analytic.

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}}$$

PS1 (Practice Problem). Show that the two following functions are analytic, i.e., the Cauchy-Riemann conditions are satisfied: $f(z) = z$ and $f(z) = z^2$.

S3. Cauchy Integral Formula. Can we evaluate a closed integral where denominator is zero inside the path region?



Let's try it with this integral below.

$$I = \oint \frac{1}{z-a} dz$$

Note that we are to go counterclockwise. But our integrand blows up at $z = a$. Strictly speaking, our integrand is not defined at this point. We refer to this as a singularity.

But our path is of integration is away from this singularity. Let's try to integrate. Make the following substitution of variables.

$$z \equiv x + iy = a + R\cos\theta + iR\sin\theta$$

Then, $z = a + \mathrm{Re}^{i\theta}$, which is equivalent to $z - a = \mathrm{Re}^{i\theta}$.

$$dz = iR\mathrm{e}^{i\theta}\, d\theta$$

$$I = \oint \frac{1}{z-a} dz = \int_0^{2\pi} \frac{iR\mathrm{e}^{i\theta}}{\mathrm{Re}^{i\theta}} d\theta = \int_0^{2\pi} i\, d\theta = 2\pi i$$

We did it!

What about this one if $f(z)$ does not have any singularities? $I = \oint \frac{f(z)}{z-a} dz$

Let's add and subtract the same term from this to obtain

$$I = \oint \frac{f(z)}{z-a} dz = \oint \frac{f(z) - f(a)}{z-a} dz + f(a) \oint \frac{1}{z-a} dz$$

Consider the two integrals on the right side separately.

$$I_A = \oint \frac{f(z) - f(a)}{z - a}\, dz$$

$$I_B = f(a) \oint \frac{1}{z - a}\, dz = 2\pi i f(a)$$

Watch what happens to the first one. Take ds to be a differential arc length for a circumference where $z - a = \mathrm{Re}^{i\theta}$.

$$\left| \oint \frac{f(z) - f(a)}{z - a}\, dz \right| \le \oint \frac{|f(z) - f(a)|}{R}\, ds \le \frac{M}{R}(2\pi R) = 2\pi M$$

M is the maximum value of $|f(z) - f(a)|$ along the circumference of the circle.

PS2 (Practice Problem). The absolute value for a real number is the distance that number is from zero. When we apply this to complex numbers we have $|z| = |x + iy| = \sqrt{x^2 + y^2}$. Show that $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$, which we used above in our analysis.

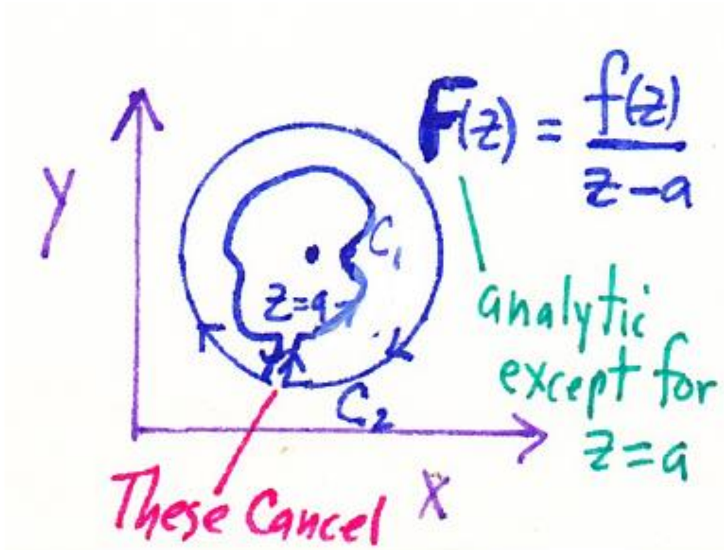
Now we make the radius R smaller and smaller.

$$\lim_{R \to 0} |f(z) - f(a)| = |f(a) - f(a)| = 0 \text{ , i.e., } \lim_{R \to 0} M = 0$$

The Cauchy Integral formula is our nice result.

$$\boxed{\oint \frac{f(z)}{z - a}\, dz = 2\pi i f(a)}$$

Our result is true for any closed path. Check out the figure and analysis below.



Start with the path $C = C_1 + C_2$ that does not contain the singularity.

$$\underbrace{\oint_{C} F(z)\,dz}_{\text{Total}} = 0$$

$$= \oint_{C_1} F(z)\,dz + \underbrace{\oint_{C_2} F(z)\,dz}_{-2\pi i f(a)} = 0$$

$$\oint_{\text{Any}} \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$