Theoretical Physics
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Chapter T Notes. Poles and the Residue Theorem

T1. Poles.

First Review:

Complex Variable: $z = x + iy$, where $\{x, y \in \mathbb{R}\}$

Complex Function: $f(z) = u(x, y) + iv(x, y)$, where $\{u, v \in \mathbb{R}\}$

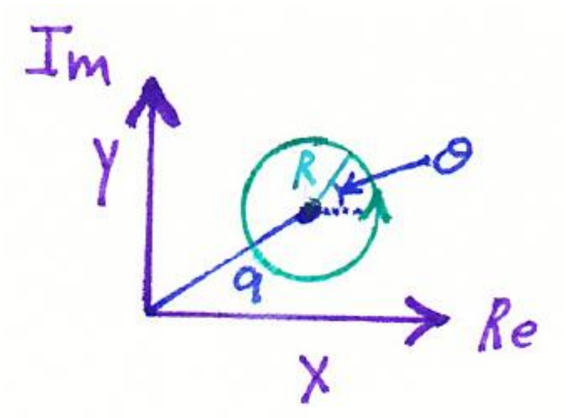
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The Cauchy-Riemann Conditions define analytic functions $f(z)$, where

$f'(z) = \dfrac{df(z)}{dz}$ is unambiguous and $\oint_C f(z)dz = 0$.

Integration is unambiguous and path independent.

Powers of z are analytic and therefore $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is analytic.



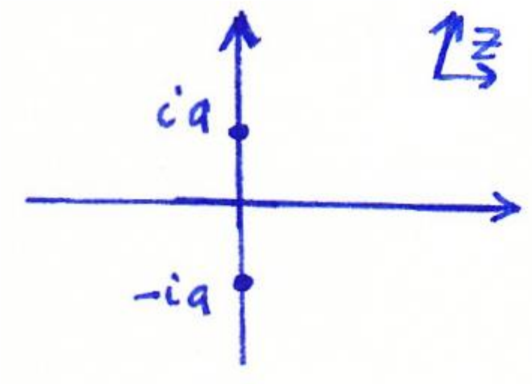
A pole a is a point where a non-analytic function "blows up." See the integrand below.

$$\oint \frac{1}{z-a} dz = 2\pi i$$

The Cauchy Integral Formula below involves an analytic function $f(z)$ divided by $z - a$, which introduces a pole for the integrand at a.

$$\boxed{\oint \frac{f(z)}{z-a} dz = 2\pi i f(a)}$$

This is true for any closed path enclosing the pole.



Find the poles for $f(z) = \dfrac{e^{imz}}{z^2 + a^2}$.

The numerator is fine. The denominator though gives us two poles. We find them by factoring and setting the denominator to zero.

$$z^2 + a^2 = (z + ia)(z - ia) = 0$$

There are two solutions that make the denominator zero. These occur for

$$z = ia \text{ and } z = -ia .$$

These are the poles. These two poles are illustrated in the above figure.

PT1 (Practice Problem). Find the poles for the following function by factoring and check your answer using the quadratic formula.

$$f(z) = \frac{z^2 + 3}{z(z^2 + iz + 2)}$$

PT2 (Practice Problem). Find the poles for the following function using the quadratic formula.

$$f(z) = \frac{e^{ikz}}{z^2 - 6z + 25}$$

T2. The Residue.

Note that for analytic functions $\oint_C f(z)dz = 0$.

For analytic functions $f(z)$, we have $\oint \frac{f(z)}{z-a} dz = 2\pi i f(a)$.

Now consider $F(z) = \frac{f(z)}{z-a}$, where $f(z)$ is analytic. The function $F(z)$ has a pole at $z-a$. We know

$$\oint F(z)\, dz = \oint \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

We can get this answer by a)clearing out the denominator of $F(z)$, b)setting $z = a$, and c)multiplying by $2\pi i$.

$$\oint F(z)\, dz = 2\pi i \left[(z-a)F(z)\right]_{z=a}$$

$$\oint F(z)\, dz = 2\pi i \left[\, f(z)\right]_{z=a} = 2\pi i f(a)$$

The value $f(a)$ is called the residue of $F(z)$ at $z = a$.

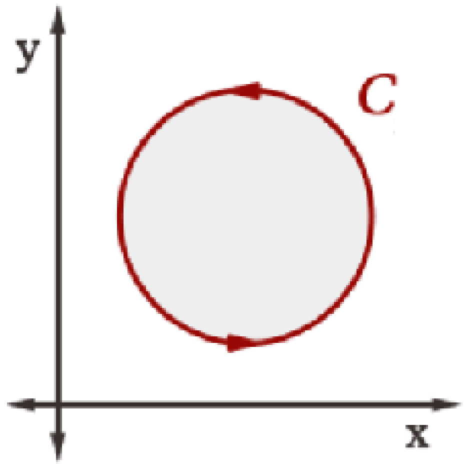
$$f(a) = Res(F, a)$$

The residue of a simple pole is given by

$$Res(F, a) = \lim_{z \to a} \left[(z-a)F(z)\right].$$

What about multiple poles? We take this up in our next section.

T3. The Residue Theorem. What about multiple poles?



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The Case of No Poles.

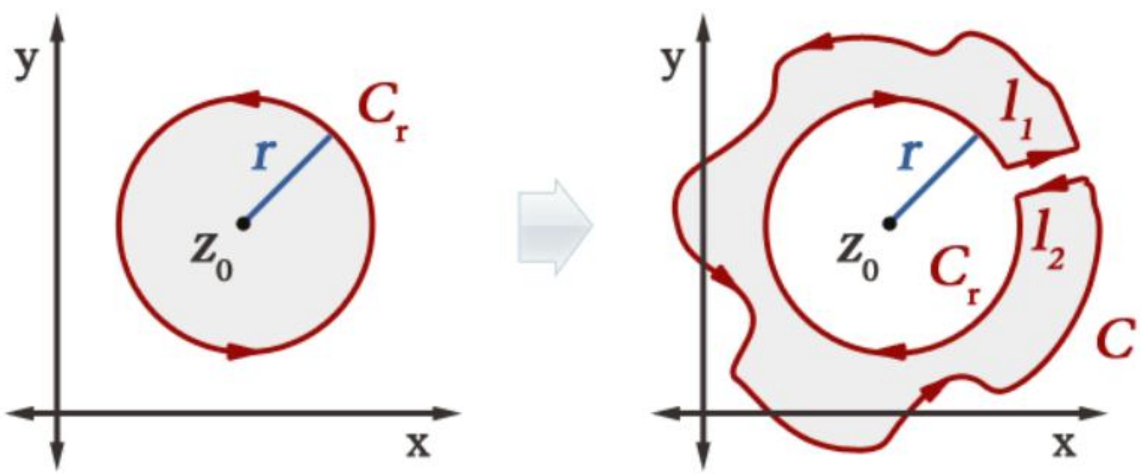
Well this is easy. A closed contour integral gives zero.

$$\oint_C F(z)dz = 0$$

The Case of One Pole. For the pole at $z = z_0$,

$$\oint F(z)\, dz = 2\pi i Res(F, z_0)$$

We emphasize below that the closed integral can be of any shape!



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$$\oint_C F(z)dz + \int_{l_2} F(z)dz + \oint_{C_r} F(z)dz + \int_{l_1} F(z)dz = 0$$

The two line integrals in the limit as the gap approaches zero sum to zero.

$$\oint_C F(z)\,dz + \oint_{C_r} F(z)\,dz = 0$$

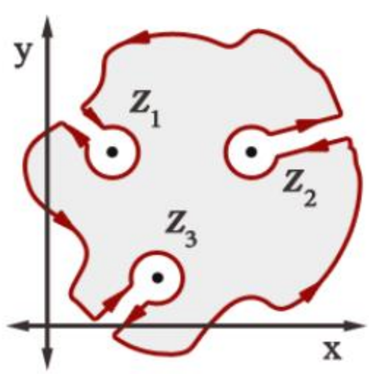
Since the second integral is clockwise, we get the following result.

$$\oint_C F(z)\,dz - 2\pi i Res(F, z_0) = 0$$

$$\oint_C F(z)\,dz = 2\pi i Res(F, z_0)$$

$$\oint_{Any} F(z)\,dz = 2\pi i Res(F, z_0)$$

The Case of Multiple Poles. For multiple poles we apply a similar trick with paths.



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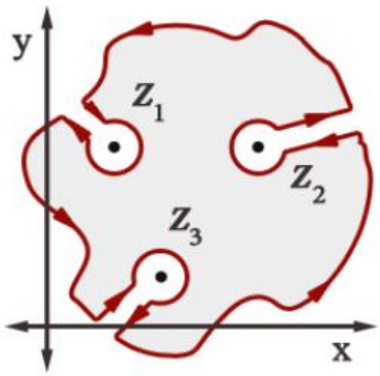
$$\oint_C F(z)\,dz - 2\pi i Res(F, z_1)$$

$$-2\pi i Res(F, z_2) - 2\pi i Res(F, z_3) = 0$$

$$\boxed{\oint_C F(z)\,dz = 2\pi i \sum_n Res(F, z_n)}$$

This is the Residue Theorem.

For our case with three poles: $$F(z) = \frac{f(z)}{(z-z_1)(z-z_2)(z-z_3)}$$



$$Res(F, z_1) = \frac{f(z_1)}{(z_1 - z_2)(z_1 - z_3)}$$

$$Res(F, z_2) = \frac{f(z_2)}{(z_2 - z_1)(z_2 - z_3)}$$

$$Res(F, z_3) = \frac{f(z_3)}{(z_3 - z_1)(z_3 - z_2)}$$

T4. Complex Integration 1. $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

We will evaluate this integral using complex variable techniques. But first, we evaluate this integral from the observation that

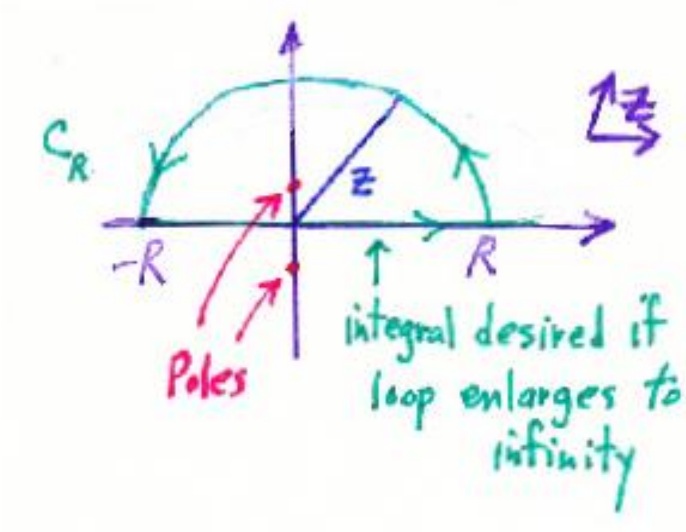
$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} .$$

So we know the answer to this one. It is always good to try a new technique on something for which we know the answer.

$$I = \tan^{-1} x \Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left[-\frac{\pi}{2} \right] = \pi$$

The method lies in this hope shown below as the radius $R \to \infty$.

$$\oint \frac{1}{1+z^2}\,dz = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} + \int_{C_R} \frac{1}{1+z^2}\,dz = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$



$$I = \oint \frac{1}{1+z^2}\,dz$$

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

Let $f(z) = \dfrac{1}{z+i}$.

$$I = \oint \frac{1}{(z+i)(z-i)}\,dz = \oint \frac{f(z)}{z-i} = 2\pi i f(i) = 2\pi i \frac{1}{2i} = \pi$$

or $$I = 2\pi i Res(F,i) = 2\pi i \left.\frac{1}{z+i}\right|_{z=i} = 2\pi i \frac{1}{2i} = \pi$$.

All we have to do now is let $R \to \infty$ and hope that the semicircle integration along C_R goes to zero. Then, the complete enclosed contour integral will give a nonvanishing answer for just the path along the complete x-axis and we are finished. You know this must happen because you have your $I = \pi$, which you know is the answer. Show

$$\lim_{R\to\infty} I_{C_R} = 0$$, where $$I_{C_R} = \int_{C_R} \frac{1}{1+z^2}\, dz$$.

$$z = Re^{i\theta}$$ and $$dz = iRe^{i\theta} d\theta$$

$$I_{C_R} = \int_{C_R} \frac{1}{1+(\mathrm{Re}^{i\theta})^2} iRe^{i\theta} d\theta$$

$$I_{C_R} = \int_{C_R} \frac{1}{1+\mathrm{R}^2\mathrm{e}^{2i\theta}} iRe^{i\theta} d\theta$$

For R large

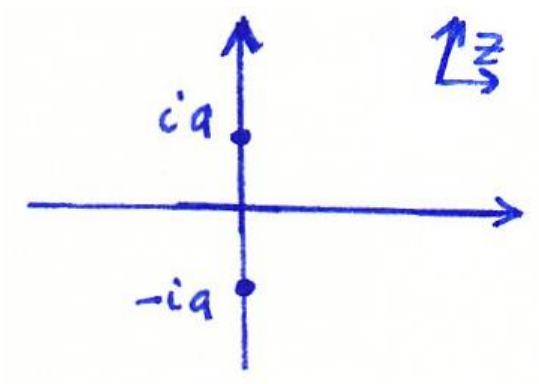
$$I_{C_R} = \lim_{R\to\infty} \int_{C_R} \frac{1}{1+\mathrm{R}^2\mathrm{e}^{2i\theta}} iRe^{i\theta} d\theta$$

$$I_{C_R} = \lim_{R\to\infty} \int_{C_R} \frac{1}{\mathrm{R}^2\mathrm{e}^{2i\theta}} iRe^{i\theta} d\theta$$

$$I_{C_R} = \lim_{R\to\infty} \int_{C_R} \frac{\mathrm{e}^{-2i\theta}}{\mathrm{R}^2} iRe^{i\theta} d\theta$$

$$I_{C_R} = i \lim_{R\to\infty} \int_{C_R} \frac{\mathrm{e}^{-i\theta}}{\mathrm{R}} d\theta = i \lim_{R\to\infty} \frac{1}{R} \int_{C_R} \mathrm{e}^{-i\theta}\, d\theta = 0$$

T5. Complex Integration 2. $I = \int_{-\infty}^{\infty} \frac{e^{imx} dx}{x^2 + a^2}$ with $a > 0$, $m > 0$



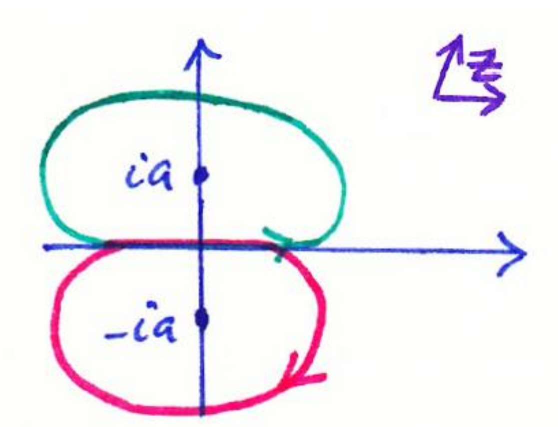
Let's organize our procedure into three steps.

Step 1. Find Your Poles.

$$F(z) = \frac{e^{imz}}{z^2 + a^2}$$

From $z^2 + a^2 = (z + ia)(z - ia) = 0$ we find two poles. $z_1 = ia$ and $z_2 = -ia$.

Step 2. Know Where to Close (Choose Your Semicircle). We need to choose the semicircle that will not mess up our vanishing semicircle result from the last section.



Which semicircle should we choose? $m > 0$

Top - along upper imaginary axis

$$iR \Rightarrow e^{im(iR)} = e^{-mR}$$

Bottom - along lower imaginary axis

$$-iR \Rightarrow e^{im(-iR)} = e^{mR}$$

The top one is the one that will vanish.

Step 3. Sum Your Residues (Use the Residue Theorem). Note that we only have one pole inside our enclosed region of integration. This is $z_1 = ia$.

$$\oint_C F(z) dz = 2\pi i Res(F, ia) \quad \text{with} \quad F(z) = \frac{e^{imz}}{(z + ia)(z - ia)}$$

$$Res(F, ia) = \left. \frac{e^{imz}}{z + ia} \right|_{z=ia} = \frac{e^{-ma}}{2ia} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{e^{imx} dx}{x^2 + a^2} = \frac{\pi}{a} e^{-ma}$$

With the Real-Imaginary Trick, we now know the two following integrals where $a > 0$ and $m > 0$.

$$I = \int_{-\infty}^{\infty} \frac{e^{imx}dx}{x^2 + a^2} = \int_{-\infty}^{\infty} \frac{\cos(mx)dx}{x^2 + a^2} + i\int_{-\infty}^{\infty} \frac{\sin(mx)dx}{x^2 + a^2}$$

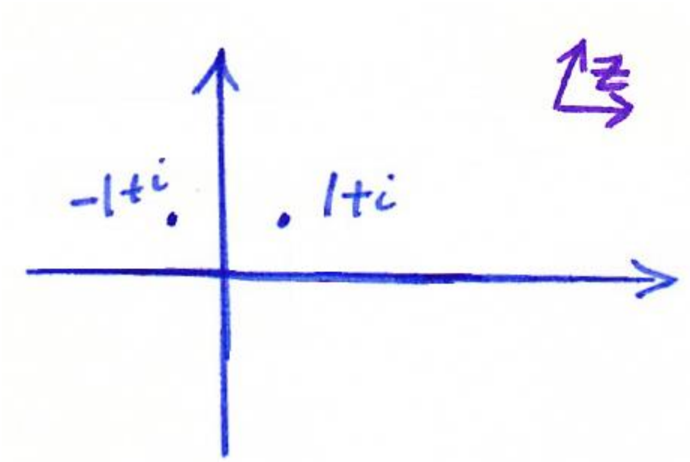
$$\int_{-\infty}^{\infty} \frac{\cos(mx)dx}{x^2 + a^2} = \frac{\pi}{a} e^{-ma} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin(mx)dx}{x^2 + a^2} = 0$$

Note that also by the symmetry argument the second integral must be zero.

T6. Complex Integration 3. $I = \int_{-\infty}^{\infty} \frac{e^{imx}dx}{x^2 - 2ix - 2}$ where $m > 0$

Step 1. Find Your Poles. $F(z) = \dfrac{e^{imz}}{z^2 - 2iz - 2}$

Use the quadratic formula with $z^2 - 2iz - 2$ to get $\dfrac{-(-2i) \pm \sqrt{(-2i)^2 - 4(1)(-2)}}{2(1)}$



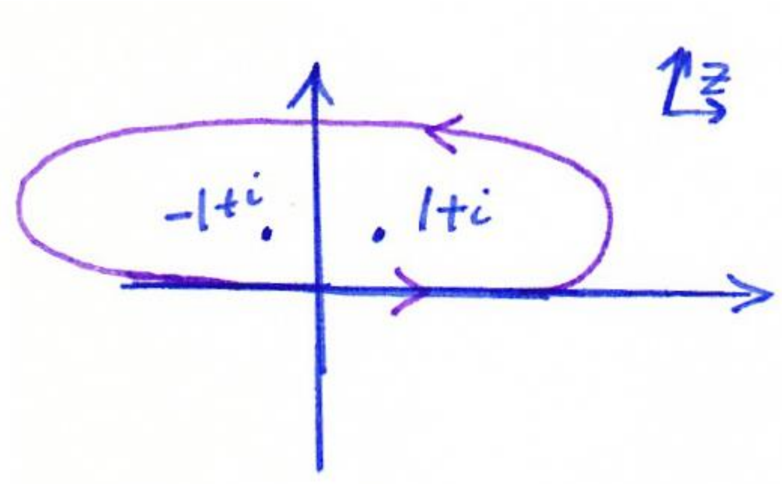
$$\frac{2i \pm \sqrt{-4+8}}{2} = \frac{2i \pm 2}{2}$$

Our poles are

$$z_1 = -1 + i$$

$$z_2 = 1 + i$$

Step 2. Know Where to Close (Choose Your Semicircle). We choose the upper plane since we have an exponential of the form e^{imz} and remember $m > 0$.

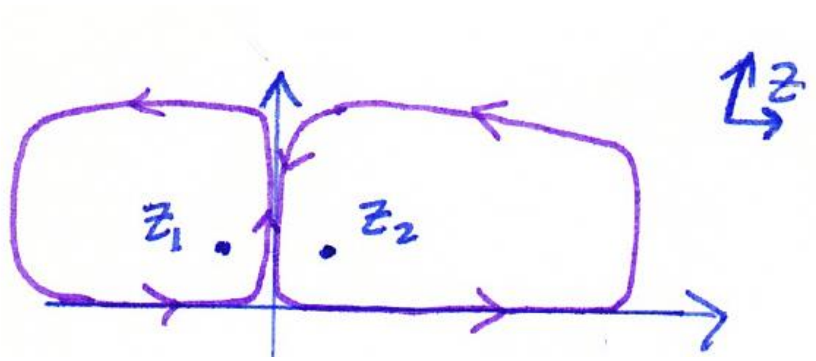


Se we close in the upper plane so that we get

$$e^{im(iR)} = e^{-mR}$$

along the imaginary axis. As $R \to \infty$ we get no trouble. Remember when we showed for a $\frac{1}{z^2}$ type of integrand that the semicircle path vanishes. We just want to make sure here that the exponential factor does not mess us up.

Step 3. Sum Your Residues (Use the Residue Theorem).



The left figure is not needed. It just reminds you about the workings of the residue theorem. So we proceed to find the residues for the poles.

$$2\pi i \sum_n Res(F, z_n)$$

$$F(z) = \frac{e^{imz}}{(z - z_1)(z - z_2)}$$ where $z_1 = -1 + i$ and $z_2 = 1 + i$

$$\sum_n Res(F, z_n) = \left. \frac{e^{imz}}{z - z_2} \right|_{z=z_1} + \left. \frac{e^{imz}}{z - z_1} \right|_{z=z_2}$$

$$\sum_n Res(F, z_n) = \frac{e^{imz_1}}{z_1 - z_2} + \frac{e^{imz_2}}{z_2 - z_1}$$ where $z_1 = -1 + i$ and $z_2 = 1 + i$

$$I = 2\pi i \sum_n Res(F, z_n)$$

$$I = \frac{2\pi i}{z_1 - z_2}\left[e^{imz_1} - e^{imz_2}\right] = \frac{2\pi i}{(-2)}\left[e^{imz_1} - e^{imz_2}\right]$$

$$I = -\pi i\left[e^{im(-1+i)} - e^{im(1+i)}\right]$$

$$I = -\pi i\left[e^{m(-i-1)} - e^{m(i-1)}\right]$$

$$I = -\pi i\left[e^{-im}e^{-m} - e^{im}e^{-m}\right]$$

$$I = -\pi i e^{-m}\left[e^{-im} - e^{im}\right]$$

$$I = \pi i e^{-m}\left[e^{im} - e^{-im}\right]$$

$$I = \pi i e^{-m}(2i)\left[\frac{e^{im} - e^{-im}}{2i}\right]$$

$$I = \int_{-\infty}^{\infty} \frac{e^{imx}dx}{x^2 - 2ix - 2} = -2\pi e^{-m} \sin m$$