

## Theoretical Physics

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### Chapter V Notes. Transfer Functions

**V1. Driven Oscillations and an LP Filter.** We return to our low-pass (LP) filter with  $R = 1$ ,  $C = 1$ , and  $f(t) = 1$  for 1 second from  $t = 0$  to  $t = 1$ . The initial charge on the capacitor is  $q(0) = 0$ . We have already solved this problem.

The charge on the capacitor is given by the following integral.

$$q(t) = \int_0^t f(u)g(t-u)du$$

This solution is the convolution of our input voltage function  $f(t)$

and the capacitor-decay response function  $g(t) = e^{-t}$ . The  $g(t)$  is our unshifted Green's function  $G(t, 0)$ . We can also write our general answer with the following Green's function notation.

$$q(t) = \int_0^t G(t, u) f(u) du \quad \text{with} \quad G(t, u) = e^{-(t-u)}$$

We would like to find the solution for a sinusoidal driving force.

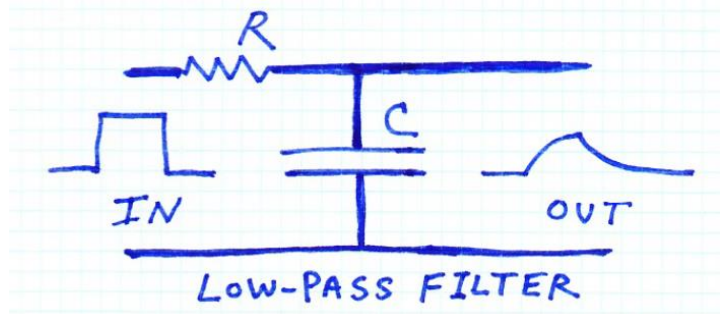
$$\text{We have } f(t) = \sin \omega t \quad \text{and} \quad G(t, u) = e^{-(t-u)}.$$

The solution is

$$q(t) = \int_0^t e^{-(t-u)} \sin(\omega u) du$$

It is easier to integrate an exponential so we use the real-imaginary trick.

$$q(t) = \text{Im} \int_0^t e^{-(t-u)} e^{i\omega u} du$$



$$q(t) = e^{-t} \operatorname{Im} \int_0^t e^u e^{i\omega u} du = e^{-t} \operatorname{Im} \int_0^t e^{(1+i\omega)u} du$$

$$q(t) = e^{-t} \operatorname{Im} \left[ \frac{e^{(1+i\omega)u}}{1+i\omega} \Big|_0^t \right]$$

$$q(t) = e^{-t} \operatorname{Im} \left[ \frac{e^{(1+i\omega)t} - 1}{1+i\omega} \right]$$

$$q(t) = e^{-t} \operatorname{Im} \left[ \frac{(e^t e^{i\omega t} - 1)(1-i\omega)}{(1+i\omega)(1-i\omega)} \right]$$

$$q(t) = e^{-t} \operatorname{Im} \left[ \frac{(e^t e^{i\omega t} - 1)(1-i\omega)}{1+\omega^2} \right]$$

$$q(t) = \operatorname{Im} \left[ \frac{(e^{i\omega t} - e^{-t})(1-i\omega)}{1+\omega^2} \right]$$

$$q(t) = \operatorname{Im} \left[ \frac{e^{i\omega t}}{1+\omega^2} (1-i\omega) \right] - \operatorname{Im} \left[ \frac{e^{-t}}{1+\omega^2} (1-i\omega) \right]$$

Taking the imaginary part of the second term we have the following.

$$q(t) = \operatorname{Im} \left[ \frac{e^{i\omega t}}{1+\omega^2} (1-i\omega) \right] + \frac{\omega e^{-t}}{1+\omega^2}$$

Now we proceed to work on the first term with help from the Euler formula.

$$q(t) = \text{Im} \left[ \frac{(\cos \omega t + i \sin \omega t)}{1 + \omega^2} (1 - i\omega) \right] + \frac{\omega e^{-t}}{1 + \omega^2}$$

$$q(t) = \frac{(\sin \omega t - \omega \cos \omega t)}{1 + \omega^2} + \frac{\omega e^{-t}}{1 + \omega^2} \quad \text{Note that } q(0) = 0.$$

Check out the second part. It is the transient component. It goes to zero as  $t$  goes to infinity. We are interested in the steady-state solution so we throw this last term away. Then,

$$q(t) = \frac{\sin \omega t - \omega \cos \omega t}{1 + \omega^2} \quad \text{for the steady state and } q(t)_{\text{transient}} = \frac{\omega e^{-t}}{1 + \omega^2}$$

The dimensions do not look correct in these equations because we have let  $R = 1$  and  $C = 1$ . We can make things look dimensionally correct by a nice trick. We know  $RC$  has dimensions of time since we saw its appearance earlier in the exponent as  $e^{-\frac{t}{RC}}$ . Therefore we can define  $\omega_c = \frac{1}{RC} = 1$  and use it as needed to get the right dimensions for frequency. The "c" in the subscript stands for cut-off. To make

$$q(t) = \frac{\sin \omega t - \omega \cos \omega t}{1 + \omega^2}$$

look dimensionally correct, we need  $\omega_c = \frac{1}{RC} = 1$  in front of the sine and a  $\omega_c^2$  in the denominator.

$$q(t) = \frac{\omega_c \sin \omega t - \omega \cos \omega t}{\omega_c^2 + \omega^2}$$

Now it looks more friendly dimensionally by inspection. Now watch this next trick to show that we have a sinusoidal oscillation that is out of phase with respect to our voltage driving function  $f(t) = \sin \omega t$ .

Do you remember the following trig identity we derived in the first chapter?

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

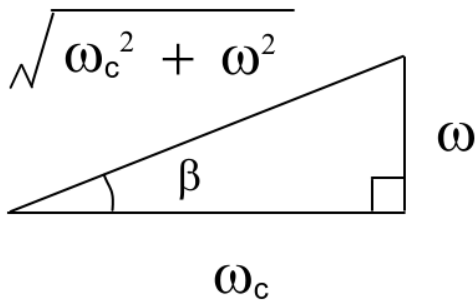
With  $\alpha = \omega t$  we have  $\sin(\omega t - \beta) = \sin \omega t \cos \beta - \cos \omega t \sin \beta$

Compare the following two equations.

$$q(t) = \frac{\omega_c \sin \omega t - \omega \cos \omega t}{\omega_c^2 + \omega^2}$$

$$\sin(\omega t - \beta) = \sin \omega t \cos \beta - \cos \omega t \sin \beta$$

Now check out the magic triangle below.



$$\omega_c = \sqrt{\omega^2 + \omega_c^2} \cos \beta$$

$$\omega = \sqrt{\omega^2 + \omega_c^2} \sin \beta$$

Then we have  $\tan \beta = \frac{\omega}{\omega_c}$  and the following.

$$q(t) = \frac{\sqrt{\omega^2 + \omega_c^2} \cos \beta \sin \omega t - \sqrt{\omega^2 + \omega_c^2} \sin \beta \cos \omega t}{\omega^2 + \omega_c^2}$$

$$q(t) = \frac{\sin \omega t \cos \beta - \cos \omega t \sin \beta}{\sqrt{\omega^2 + \omega_c^2}}$$

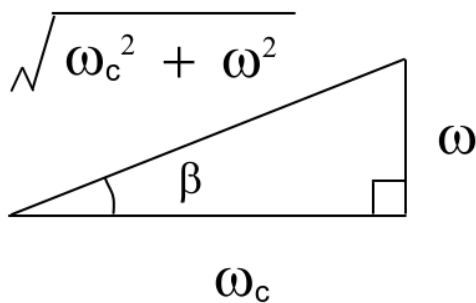
$$q(t) = \frac{\sin(\omega t - \beta)}{\sqrt{\omega^2 + \omega_c^2}}$$

With  $V_{in}(t) = f(t) = \sin \omega t = V_0 \sin \omega t$

$$V_{out}(t) = \omega_c V_0 \frac{\sin(\omega t - \beta)}{\sqrt{\omega^2 + \omega_c^2}} \quad \text{Why the } \omega_c \text{ out in front?}$$

Hint: Dimensional Analysis!

Comparing the amplitudes, we obtain



$$T \equiv \left| \frac{V_{out}(t)}{V_{in}(t)} \right| = \frac{\omega_c}{\sqrt{\omega^2 + \omega_c^2}}$$

where the output has phase given by

$$\tan \beta = \frac{\omega}{\omega_c}$$

Now you can see below why this is a low-pass filter.

Low Frequencies:  $\lim_{\omega \rightarrow 0} \left| \frac{V_{out}}{V_{in}} \right| = \lim_{\omega \rightarrow 0} \frac{\omega_c}{\sqrt{\omega^2 + \omega_c^2}} = \frac{\omega_c}{\omega_c} = 1$

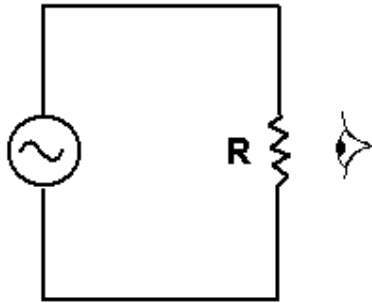
High Frequencies:  $\lim_{\omega \rightarrow \infty} \left| \frac{V_{out}}{V_{in}} \right| = \lim_{\omega \rightarrow \infty} \frac{\omega_c}{\sqrt{\omega^2 + \omega_c^2}} = 0$

One final note, electrical engineers prefer the following conventions.

$$V_{in}(t) = V_0 \sin \omega t \quad \text{and} \quad V_{out}(t) = \omega_c V_0 \frac{\sin(\omega t + \phi)}{\sqrt{\omega^2 + \omega_c^2}}$$

$$\phi = -\beta \quad \text{and} \quad \tan \phi = \tan(-\beta) = -\tan \beta = -\frac{\omega}{\omega_c}$$

**V2. Phasors.** Engineers often just need  $\left| \frac{V_{out}}{V_{in}} \right|$  and the phase angle  $\phi$ . The phasor was developed to arrive at these quickly.

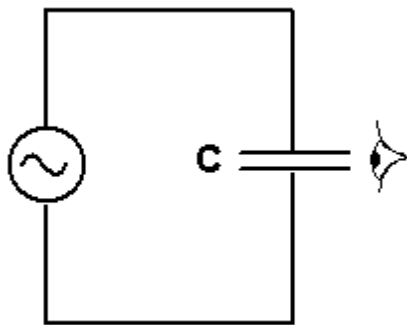


$$V = IR \quad \text{with input voltage } V(t) = V_0 \cos \omega t$$

$$V_0 \cos \omega t = I(t)R$$

The current is in phase with the input voltage source.

$$I(t) = \frac{V_0 \cos \omega t}{R}$$

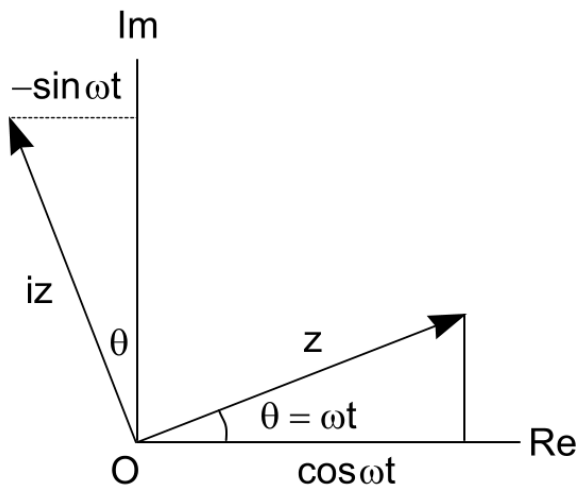


$$V = \frac{q}{C} \quad \text{with input voltage } V(t) = V_0 \cos \omega t$$

$$V_0 \cos \omega t = \frac{q(t)}{C} \quad q(t) = CV_0 \cos \omega t$$

$$i(t) = \frac{dq(t)}{dt} = -\omega CV_0 \sin \omega t$$

The current is not in phase with the input voltage source.



$$\text{Applied Voltage: } V(t) = V_0 \cos \omega t$$

$$\text{Current: } i(t) = -\omega CV_0 \sin \omega t$$

See the figure. The current leads the voltage by 90 degrees. The lines with arrows in the figure are called phasors.

You use the real part of the phasor for the applied voltage and current.

Remember our trick to rotate by 90 degrees? You multiply a complex number  $z$  by  $i$ , "making a left face." So in "phasor" language we have the following.

Phasor  $V(t) = V_0 e^{j\omega t}$ , where  $j = \sqrt{-1}$  because  $i$  is reserved for current.

It is understood you take the real part:  $V(t) = V_0 \operatorname{Re} e^{j\omega t} = V_0 \cos \omega t$

Phasor Current  $i(t) = j\omega CV(t)$

Check:  $i(t) = \operatorname{Re} [j\omega CV(t)] = \operatorname{Re} [j\omega CV_0 e^{j\omega t}] = -\omega CV_0 \sin \omega t$

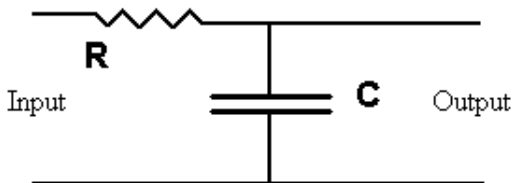
So in "phasor" language we write:  $i(t) = j\omega CV(t)$

Comparing this to Ohm's Law:  $i(t)R = V(t)$ , the effective resistance depends on frequency. We call this the impedance.

$$V(t) = i(t)Z \quad \text{and} \quad i(t) = V(t) / Z, \quad \text{where} \quad Z = \frac{1}{j\omega C}.$$

**V3. Transfer Function.** Engineers are usually interested in the relative

transmission amplitude  $T = \left| \frac{V_{out}(t)}{V_{in}(t)} \right|$  through the filter and the phase  $\phi$ . Watch this powerful method using phasors. But first, recall the answer we obtained using the Green's function:



$$T = \left| \frac{V_{out}(t)}{V_{in}(t)} \right| = \frac{\omega_c}{\sqrt{\omega^2 + \omega_c^2}}$$

$$\tan \phi = -\frac{\omega}{\omega_c}$$

We write the regular voltage equation with impedances. Note that the impedance for the resistor is simply  $R$  since there is no phase shift for the resistor.

Phasors:  $V_{in}(t) = i(t)Z_R + i(t)Z_C$  and  $V_{out}(t) = i(t)Z_C$

$$V_{in}(t) = i(t) \left[ R + \frac{1}{j\omega C} \right] \quad \text{and} \quad V_{out}(t) = i(t) \frac{1}{j\omega C}$$

Transfer Function

$$H(\omega) = \frac{V_{out}(t)}{V_{in}(t)} = \frac{1/(j\omega C)}{R + \frac{1}{j\omega C}} \quad H(\omega) = \frac{1}{1 + j\omega RC}$$

The Magnitude of the Transfer Function is our Transmission

$$T = |H(\omega)| = \left| \frac{1}{1 + j\omega RC} \right|$$

**PV1 (Practice Problem).** From  $|z| = |x + iy| = \sqrt{x^2 + y^2}$ , show that  $\left| \frac{1}{z} \right| = \frac{1}{|z|}$ .

$$T = |H(\omega)| = \frac{1}{\sqrt{1 + \omega^2 (RC)^2}}$$

Remember our basic definition from before:  $\omega_c = \frac{1}{RC}$  (the cutoff frequency).

$$T = |H(\omega)| = \frac{1}{\sqrt{1 + \omega^2 / (\omega_c)^2}}$$

$$T = \left| \frac{V_{out}(t)}{V_{in}(t)} \right| = \frac{\omega_c}{\sqrt{\omega^2 + \omega_c^2}}$$

This is our answer from before.

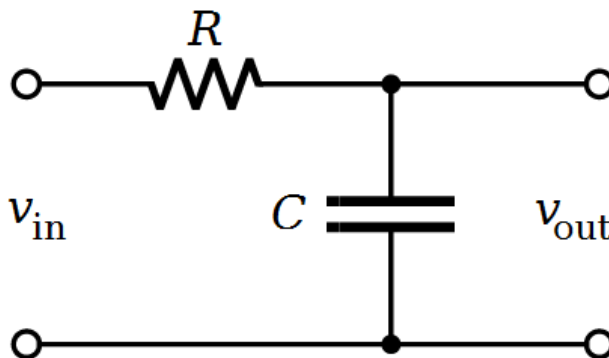


From  $H(\omega) = \frac{1}{1 + j\omega RC} = \frac{1 - j\omega RC}{1 + (\omega RC)^2}$  we get the phase.

$$\tan \phi = \frac{\text{Im } H(\omega)}{\text{Re } H(\omega)} \quad \tan \phi = \frac{-\omega RC}{1} \quad \tan \phi = -\frac{\omega}{\omega_c}$$

Summary:  $H(\omega) = \frac{1}{1 + j\omega RC}$      $T = |H(\omega)| = \frac{1}{\sqrt{1 + \omega^2 / (\omega_c)^2}}$

$$V_{in}(t) = V_0 \sin \omega t \quad \text{and} \quad V_{out}(t) = \omega_c V_0 \frac{\sin(\omega t + \phi)}{\sqrt{\omega^2 + \omega_c^2}}$$



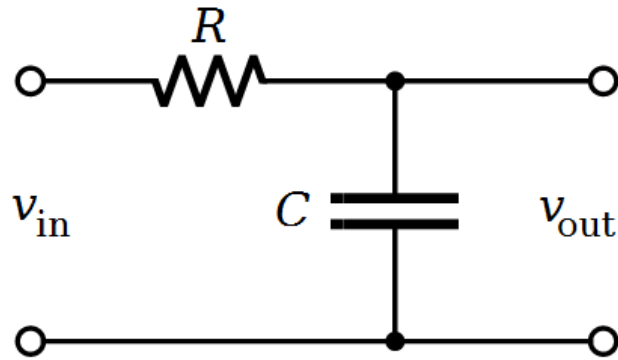
Courtesy Inductiveload, Wikimedia Commons

The phase of the voltage across the capacitor is behind since  $\tan \phi = -\frac{\omega}{\omega_c} < 0$ .

The phase ranges as follows:  $-\frac{\pi}{2} \leq \phi \leq 0$ .

**PV2 (Practice Problem).** Sketch  $T(\omega) = |H(\omega)|$  and  $\phi = -\tan^{-1} \frac{\omega}{\omega_c}$  for all input frequencies  $\omega$ . At which frequencies is the phase 0 and  $-\pi / 2$ ?

#### V4. Transfer Function and Transforms.



$$v_{in}(t) = \frac{dq}{dt} R + \frac{q}{C} \qquad v_{out}(t) = \frac{q}{C}$$

$$\mathfrak{T}\{V_{in}(t)\} = (j\omega)RQ(\omega) + \frac{1}{C}Q(\omega) \qquad \mathfrak{T}\{V_{out}(t)\} = \frac{1}{C}Q(\omega)$$

$$\frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{Q(\omega)/C}{j\omega RQ(\omega) + Q(\omega)/C} = \frac{1}{1 + j\omega RC}$$

$$\frac{V_{out}(\omega)}{V_{in}(\omega)} = H(\omega)$$

In transform space, the ratio of the Fourier transform of the output to the Fourier transform of the input is the transfer function!

**PV3 (Practice Problem).** Show that the above can be arrived at from the Laplace Transform if you take  $q(0) = 0$  and make the substitution  $s = j\omega$  after you take the Laplace Transform.