Theoretical Physics **Exam 3** **April 19, 2021**
Prof. Ruiz, UNC Asheville

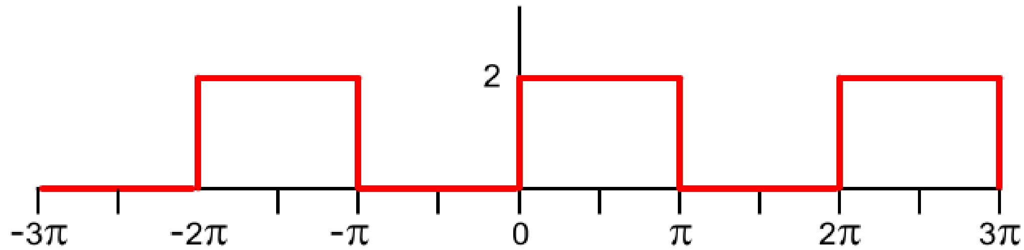
Open Book/Notes/Internet (90 minutes including scan and email return).

Posted Syllabus Exam Time is 6:00 pm – 7:30 pm.

Exam emailed 5:45 pm and you can start as soon as you get it.

All work and related steps must be explicitly shown for full credit.

[25] 1. Fourier Series. Find the Fourier Series for the periodic train of pulses shown below.



For full credit, do all integrals and write out your answer by giving the first five nonzero terms in its simplest and most elegant form. You must write f(x) = and then give the coefficients multiplied by the appropriate trig function for 5 nonzero terms where each coefficient is in simplest mathematical form. The first term might be a constant.

[25] 2. Laplace Transform. Find the Laplace transform of $f(t) = t^n$, where n = 0, 1, 2, 3 … and you employ the derivative trick in evaluating the integral for the general case for n > 0.

[15] 3. Convolution. Find the convolution $f(t) * g(t)$ where $f(t) = t$ and $g(t) = t$, i.e., find the convolution $f(t) * g(t) = t * t$.

[10] 4. Cauchy-Riemann Conditions. You have a function $f(x, y) = u(x, y) + v(x, y)i$, where $u(x, y)$ and $v(x, y)$ are real functions. If $u(x, y) = x$, give a function $v(x, y)$ so that the Cauchy-Riemann conditions are NOT met.

[25] 5. Complex Integration. Evaluate $I(t) = \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{\omega^2 - i\omega + 2} d\omega$ using complex integration. Include a figure showing the contour integration used and the location of all poles. You do not need to prove that the circular portion of the contour vanishes, but, you need to justify why you chose your particular circular path.

1. Fourier. $a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx = \frac{1}{\pi}2\pi = 2 \qquad a_n = \frac{1}{\pi}\int_0^{\pi} 2\cos(nx)dx = \frac{2}{\pi}\frac{\sin(nx)}{n}\Big|_0^{\pi} = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi}\sin(nx)dx = \frac{2}{\pi}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi} = \frac{2}{\pi n}[1-\cos(n\pi)] = \frac{4}{\pi n} \text{ for odd } n$$

$$f(x) = 1 + \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x)\ldots\right]$$

Check: Lift up the square wave we did in class by adding 1 and you get the above answer.

2. Laplace. $F(s) = \int_0^{\infty} t^n e^{-st}dt = \left[-\frac{d}{ds}\right]^n \int_0^{\infty} e^{-st}dt = \left[-\frac{d}{ds}\right]^n L\{1\}$

$$L\{1\} = \frac{e^{-st}}{-s}\Big|_0^{\infty} = 0 - \left[\frac{1}{-s}\right] = \frac{1}{s} \qquad F(s) = \left[-\frac{d}{ds}\right]^n \frac{1}{s} = \frac{n!}{s^{n+1}}$$

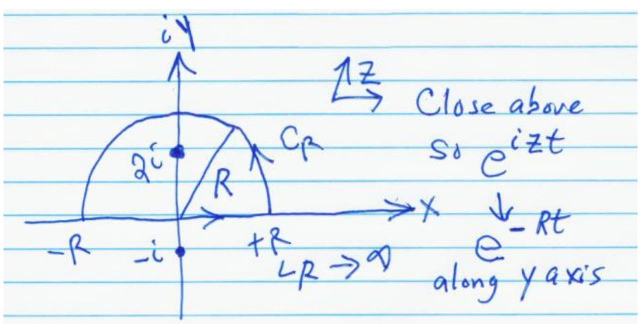
3. Convolution. $f(t) * g(t) = t * t = \int_0^t f(u)g(t-u)du = \int_0^t u(t-u)du$

$$t * t = t\int_0^t u\,du - \int_0^t u^2 du = t\frac{u^2}{2}\Big|_0^t - \frac{u^3}{3}\Big|_0^t = \frac{t^3}{2} - \frac{t^3}{3} = \frac{t^3}{6}$$

4. Cauchy-Riemann. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ with $f(x,y) = x + v(x,y)i$

Since $\frac{\partial u}{\partial x} = 1$, we can go with $v(x,y) = xy$ so that $\frac{\partial v}{\partial y} = x \neq \frac{\partial u}{\partial x}$.

5. Complex Integration.



$$I(t) = \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{(\omega+i)(\omega-2i)}d\omega$$

$$I(t) = \oint \frac{e^{izt}}{(z+i)(z-2i)}dz$$

$$I(t) = 2\pi i Res\left[\frac{e^{izt}}{(z+i)(z-2i)}, 2i\right] \qquad I(t) = 2\pi i\frac{e^{izt}}{(z+i)}\Big|_{z=2i} = \frac{2\pi}{3}e^{-2t}$$