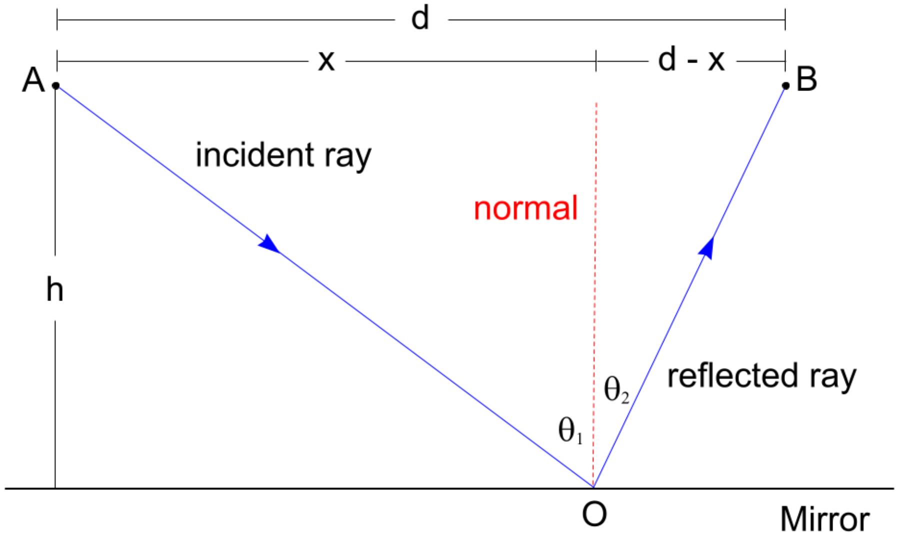
Modern Optics, Prof. Ruiz, UNC Asheville
Chapter A Solutions. The Principle of Least Time

HW-A1. Law of Reflection. Derive the Law of Reflection from the Principle of Least Time. A light ray leaves point A to reflect off a mirror and arrive at point B. Find an equation for the time t(x) as a function of x for the trip from A to B, where the speed of light is given by c. Minimize the time and show that x = d / 2 and that the angle of incidence θ_1 equals the angle of reflection θ_2.



Neat work including diagram (no crossing out) 4 pts, calculus 4 pts, explain conclusion 2 pts.

$$t = t_{\mathrm{AO}} + t_{\mathrm{OB}}$$

$$t = \frac{d_{\mathrm{AO}}}{c} + \frac{d_{\mathrm{OB}}}{c}$$

$$t = \frac{\sqrt{x^2 + h^2}}{c} + \frac{\sqrt{(d-x)^2 + h^2}}{c}$$

To find the path with the minimum time: $\dfrac{dt}{dx} = 0$

$$\frac{dt}{dx} = \frac{1}{2}\frac{2x}{c\sqrt{x^2 + h^2}} + \frac{1}{2}\frac{(d-x)(-1)}{c\sqrt{(d-x)^2 + h^2}} = 0$$

$$\frac{x}{\sqrt{x^2 + h^2}} + \frac{(d-x)(-1)}{\sqrt{(d-x)^2 + h^2}} = 0$$

$$\frac{x}{\sqrt{x^2+h^2}} = \frac{d-x}{\sqrt{(d-x)^2+h^2}}$$

Method 1.

$\sin\theta_1 = \sin\theta_2$ gives $\theta_1 = \theta_2$, leading to $x = \dfrac{d}{2}$.

Method 2. Or, you can continue as follows instead.

$$\frac{x^2}{x^2+h^2} = \frac{(d-x)^2}{(d-x)^2+h^2}$$

$$x^2\left[(d-x)^2+h^2\right] = (d-x)^2(x^2+h^2)$$

$$x^2(d-x)^2 + x^2h^2 = (d-x)^2x^2 + (d-x)^2h^2$$

$$x^2h^2 = (d-x)^2h^2$$

$$x^2 = (d-x)^2$$

$$x = \pm(d-x)$$

$x = (d-x)$, positive square root since $x > 0$

$2x = d$ and $x = \dfrac{d}{2}$

$\tan\theta_1 = \dfrac{x}{h} = \dfrac{d}{2h}$ and $\tan\theta_2 = \dfrac{d-x}{h} = \dfrac{d-(d/2)}{h} = \dfrac{d}{2h}$

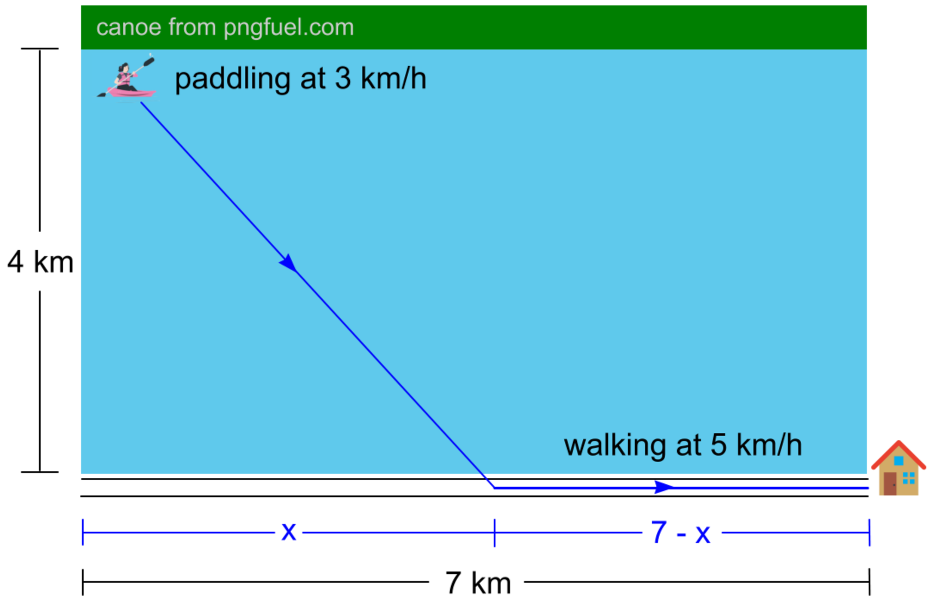
$$\tan\theta_1 = \tan\theta_2$$

Since the angles are both less than 90° we are thus in the first quadrant in trigonometry. Therefore, we must have

$$\theta_1 = \theta_2$$

$$\theta_{\text{incidence}} = \theta_{\text{reflection}}$$

HW-A2. Getting Home in the Least Time. A lady is paddling in a canoe at 3 km/h and then will walk the rest of the way home at 5 km/h. For the shortest trip, what should x be when she touches land? Always include units with answers. Neglect the width of the street. Give the total time for the fastest trip and the times for the two longer trips where x = 0 and x = 7 km. First give the answers as fractions and/or radicals and then convert to decimal with three significant figures, e.g., $\sqrt{3}+2=3.73\text{ hours}$. Neat with diagram 4 pts, 2 pts for each of the three correct fraction/radical answers.



$$t = t_{\text{paddling}} + t_{\text{walking}}$$

$$t = \frac{d_{\text{paddling}}}{v_{\text{paddling}}} + \frac{d_{\text{walking}}}{v_{\text{walking}}}$$

$$t(x) = \frac{\sqrt{x^2+4^2}}{3} + \frac{7-x}{5} = \frac{\sqrt{x^2+16}}{3} + \frac{7-x}{5}$$

$$\frac{dt(x)}{dx} = \frac{1}{2}\frac{2x}{3\sqrt{x^2+16}} - \frac{1}{5} = 0$$

$$\frac{x}{3\sqrt{x^2+16}} = \frac{1}{5}$$

$$5x = 3\sqrt{x^2+16}$$

$$25x^2 = 9(x^2+16)$$

$$25x^2 = 9x^2 + 9 \cdot 16$$

$$16x^2 = 9 \cdot 16$$

$$x^2 = 9$$

$$x = 3 \text{ km}$$

$$t(3) = \frac{\sqrt{x^2 + 16}}{3} + \frac{7 - x}{5}$$

$$t = \frac{\sqrt{3^2 + 16}}{3} + \frac{7 - 3}{5}$$

$$t = \frac{\sqrt{9 + 16}}{3} + \frac{4}{5}$$

$$t = \frac{\sqrt{25}}{3} + \frac{4}{5}$$

$$t = \frac{5}{3} + \frac{4}{5}$$

$$t = \frac{25 + 12}{15}$$

$$t(3) = \frac{37}{15} = 2.47 \text{ hours}$$

$$t(0) = \frac{\sqrt{0^2 + 16}}{3} + \frac{7 - 0}{5} = \frac{\sqrt{16}}{3} + \frac{7}{5} = \frac{4}{3} + \frac{7}{5} = \frac{20 + 21}{15} = \frac{41}{15} = 2.73 \text{ hours}$$

$$t(7) = \frac{\sqrt{7^2 + 16}}{3} + \frac{7 - 7}{5} = \frac{\sqrt{49 + 16}}{3} + 0 = \frac{\sqrt{65}}{3} = 2.69 \text{ hours}$$