B0. Introduction

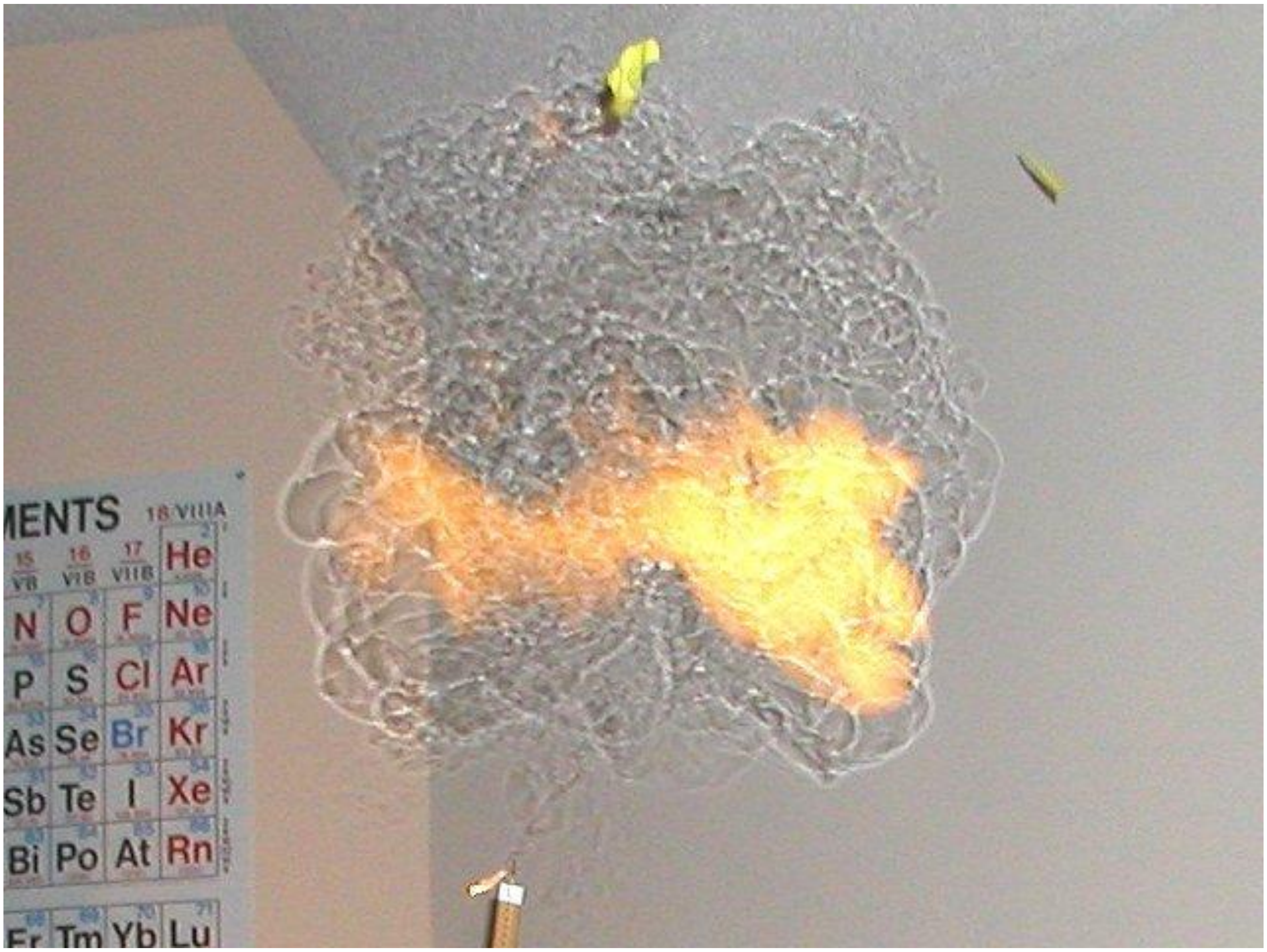

Fig. 0. Shimmering due to variation in air index of refraction due to heat.
Exploding a hydrogen balloon in astronomy. Dr. Booker led the demo. Photo by Doc. Robinson Hall 125, UNC Asheville, February 28, 2002.

We learn about refraction and the index of refraction in this chapter. The basic law of refraction is commonly known as Snell's Law. In Fig. 0 a hydrogen balloon was exploded safely under the direction of Dr. Booker. DO NOT attempt this experiment at home since hydrogen gas is very explosive. A hydrogen balloon was prepared by lab manager Susie Wright as she ripped off some hydrogen from a gas tank in chemistry. She brought the hydrogen balloon to class attached to a long string. Dr. Booker lit a match attached to a meter stick and had a student safely light the balloon with the meter stick. The balloon exploded and I was ready snapping a photo. The ensuing heat in nonuniform areas caused the light to shimmer. Light passing through nonuniform temperatures produce mirages, a topic included in this chapter. You can consider the shimmering effect in Fig. 0 a mirage.

B1. Snell's Law (High School Derivation)

Before we proceed to the sophisticated derivation of Snell's Law using calculus and the Principle of Least Time, let's review a basic derivation from introductory physics. This approach does not involve calculus, just trigonometry. Therefore, it is taught in high school courses in physics.

I recall seeing the derivation in high school with a diagram similar to Fig. 1 below, but without marching soldiers. We are going to include the marching soldiers. I first learned about the marching soldiers model many years ago from one of my mentors, Dr. Richard Berg, when I was a graduate student at the University of Maryland. He was teaching a course called *The Physics of Sound and Music*. I brought this course to UNCA (c. 1980) and developed an online course delivery system for it with my computer programming son Evan when he was in high school (1998). This course-delivery system became a top story on CNN in 2002: http://www.mjtruiz.com/television.php

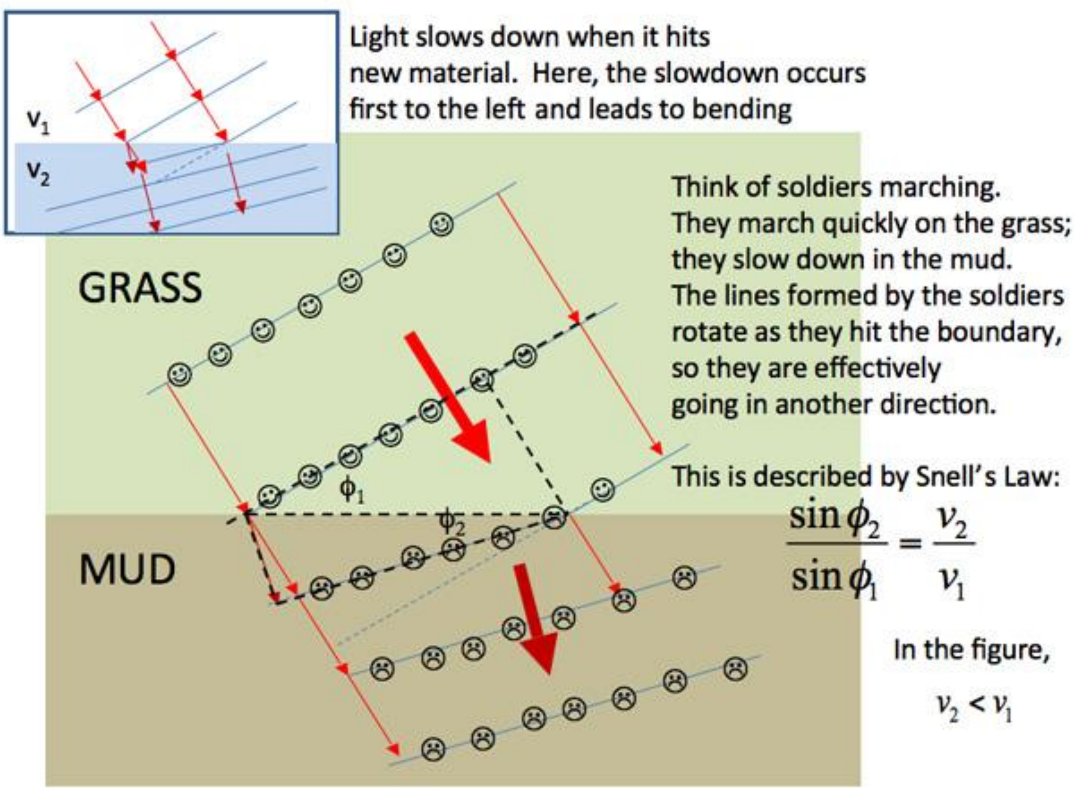

Fig. 1. Refraction and Marching Soldiers. Image Courtesy University Corporation for Atmospheric Research (NCAR), Boulder, Colorado, Material Supported by the National Science Foundation (NSF) and NCAR.

A nice figure of the marching-soldiers model appears in Fig. 1 Courtesy the University Corporation for Atmospheric Research in Boulder Colorado. Soldiers march at an angle entering mud from grass. They slow down in the mud. But conservation of people requires that the number of rows passing an observer every minute must be the same in the grass region and the mud region. Soldiers are conserved. This fact means that the frequency of rows passing by per minute remains the same in the grass and mud. But the distance between the rows decreases in the mud as the speed of the marching soldiers decreases. You can't march in mud as fast as you can in the grass.

Let's define some important quantities here.

1. The wavelength λ – the distance from one row to the next.
2. The frequency f – the number of rows that pass you each second (will be a fraction).
3. The speed v – the speed of the marching soldiers.
4. The period T – the time for one row to move a distance of one wavelength.

The speed, which is distance / time is then given as

$$v = \frac{\lambda}{T} \, .$$

But the period T has an inverse relation with frequency f. If one row passes you in 10 seconds, then the frequency of the rows going by you is 1/10 per second. Therefore,

$$f = \frac{1}{T} \, ,$$

and the speed can also be expressed as

$$v = \lambda f \, .$$

Now we are ready to derive Snell's law. See Fig. 2 below, one now looking extremely similar to the diagram in my high school physics book. I still possess a copy of the book which I purchased for my senior year (1967-1968) at Bishop Eustace Prep in Pennsauken, NJ: *Physics* (2^{nd} edition) by the Physical Science Study Committee (D. C. Heath and Company, Boston, 1966).

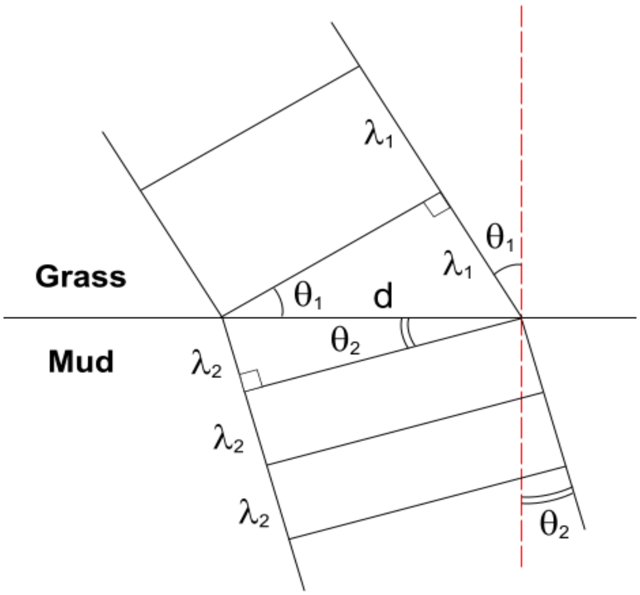

Fig. 2. Refracted Wave Crests of Marching Soldiers.

The crests change direction as the soldiers first hit the mud on the left extreme of the row. They start to march slower while those on the right extreme are still marching faster. So the row undergoes a change in direction much like a car swerving as it enters mud at an angle.

Since $v = \lambda f$ is true in the grass and in the mud, while the frequency remains the same due to conservation of people (rows passing by per second), the wavelength shortens when the speed decreases. So we can write the following pair of equations,

$$v_1 = \lambda_1 f \quad \text{and} \quad v_2 = \lambda_2 f \, .$$

These equations lead to $\dfrac{v_1}{\lambda_1} = \dfrac{v_2}{\lambda_2}$. But $\lambda_1 = d \sin\theta_1$ and $\lambda_2 = d \sin\theta_2$. Therefore,

Fig. 3. The traveling ants again.

$\dfrac{v_1}{d \sin\theta_1} = \dfrac{v_2}{d \sin\theta_2}$ and $\dfrac{v_1}{\sin\theta_1} = \dfrac{v_2}{\sin\theta_2}$. Now use the index of refraction definition: $n = \dfrac{c}{v}$ to get $v = \dfrac{c}{n}$ and $\dfrac{c}{n_1 \sin\theta_1} = \dfrac{c}{n_2 \sin\theta_2}$, which means

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \, .$$

The ants figured it out. See Fig. 3, the same figure we encountered in the previous chapter. Note how the ants travel approximately in a straight line in the white region as they approach the green region. Then they make a bend downward and travel approximately in a straight line in the slower-traveling green region. The green is some sort of green mesh or grass, on which the ants cannot walk as fast.

What we know today as Snell's Law can historically be traced earlier through three key scientists shown below. Ptolemy, the Greek mathematician and astronomer, discovered a refraction relation that was true only for small angles. The Arab mathematician, astronomer, and physicist al-Haytham, also known by the Latinized form Alhazen, wrote his *Book of Optics*, rich in the physics of optics with a version of the law of refraction close to what we know today. However, the Persian Ibn Sahl arrived at the law and used it his designs of lenses.

		
Claudius Ptolemy (c. 100 – 170)	Hasan Ibn al-Haytham (c. 965 – c. 1040)	Ibn Sahl (c. 940-1000)
Greek Mathematician and Astronomer	Arab Mathematician, Astronomer and Physicist	Persian Mathematician and Physicist
Refraction for Small Angles	Closer in his *Book of Optics*	Discovers the Law

Fig. 4. Three scientists who worked on refraction. Ibn Sahl finds the law.
Credits: Ptolemy and al-Haytham images from School of Mathematics and Statistics Univ. of St. Andrews, Scotland; Sahl image from https://islaminindonesia.wordpress.com/

The contribution of Sahl was brought to the attention of the academic community only fairly recently in 1990 by the historian of science Roshdi Rashed. Rashed (b. 1936), also a mathematician, specializes in the ancient work of Persian and Arabic scientists. During the 1000 years of the western "Middle Ages" starting at the Fall of the Roman Empire (c. 460) European science was stagnant. This period is also called the "Dark Ages." During this time Arabic and Persian science were most prominent in the world. Today, most of our names of stars have Arabic names that date from this period.

Centuries later Europe came alive scientifically during the "Scientific Revolution" initiated by the Polish astronomer Copernicus with his famous book of 1543, published the year of his death. The law of refraction was "rediscovered" independently by the four individuals in Fig. 5: the English mathematician and astronomer Harriot, the Dutch mathematician and astronomer Snel, the French philosopher and mathematician Descartes, and the French lawyer and mathematician Fermat. The reason for so many rediscoveries is that the first two, Harriot and Snel, did not publish their work.

			
Thomas Harriot (c. 1560 – 1621)	Willebrord Snel van Royen (1580-1626)	René Descartes (1596-1650)	Pierre de Fermat (1601-1665)
English Mathematician and Astronomer	Dutch Mathematician and Astronomer	French Philosopher, Mathematician	French Lawyer and Mathematician
Rediscovery (1602)	Rediscovery (1621)	Rediscovery (1637)	Least Time (1662)

Fig. 5. Rediscoveries of the Law of Refraction
Image Credits: Wikipedia

The law is commonly known as Snell's Law or the Snell-Descartes law. Note that Snell's original name is spelled "Snel." So you can even find references to Snel's Law with one "l" in the name. Now we turn to the lawyer Fermat and the Principle of Least Time.

B2. Snell's Law (Fermat Derivation)

We now proceed to the least time calculation, one involving two media. We return to our lifeguard in Fig. 6. This analysis is very profound since it offers a connection of the ray diagrams of geometrical optics to the wave result in the previous section. We will get Snell's Law without introducing wavelength!

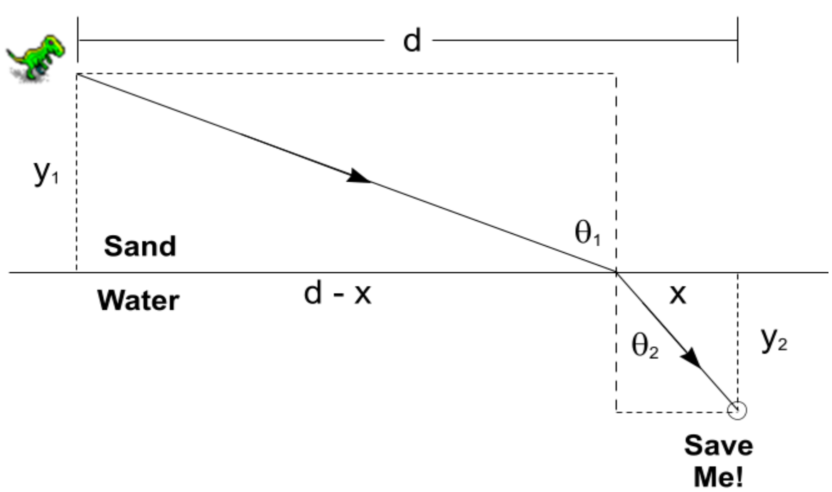

Fig. 6. A Lifeguard Getting to the Drowning Person in the Shortest Time.

In Fig. 6 the lifeguard needs to get to the person in the water as fast as possible. You can run faster than you can swim. Therefore, the shortest time is NOT the shortest distance. You run more in the sand where your speed is faster, then turn and swim at a slower speed to the victim. Then, the time to get to your destination is the sum of the time t_1 in the sand, where you travel a distance d_1 at speed v_1, and the time t_2 in the water, where you travel a distance d_2 at speed v_2. From Fig. 6,

$$t = t_1 + t_2 = \frac{d_1}{v_1} + \frac{d_2}{v_2} = \frac{\sqrt{(d-x)^2 + y_1^2}}{v_1} + \frac{\sqrt{x^2 + y_2^2}}{v_2}.$$

Now use $n_1 = c / v_1$ and $n_2 = c / v_2$ as given definitions.

$$t = \frac{n_1}{c}\sqrt{(d-x)^2 + y_1^2} + \frac{n_2}{c}\sqrt{x^2 + y_2^2}$$

To find the x that minimizes the time we set the derivative with respect to x to zero.

$$\frac{dt}{dx} = \frac{n_1}{c}\frac{1}{2}\frac{2(d-x)(-1)}{\sqrt{(d-x)^2 + y_1^2}} + \frac{n_2}{c}\frac{1}{2}\frac{2x}{\sqrt{x^2 + y_2^2}} = 0$$

$$\frac{dt}{dx} = -\frac{n_1}{c}\frac{(d-x)}{\sqrt{(d-x)^2 + y_1^2}} + \frac{n_2}{c}\frac{x}{\sqrt{x^2 + y_2^2}} = 0$$

$$\frac{n_1}{c}\frac{(d-x)}{\sqrt{(d-x)^2 + y_1^2}} = \frac{n_2}{c}\frac{x}{\sqrt{x^2 + y_2^2}}$$

$$n_1\frac{(d-x)}{\sqrt{(d-x)^2 + y_1^2}} = n_2\frac{x}{\sqrt{x^2 + y_2^2}}$$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

We have obtained the famous result!

B3. The Index of Refraction

The parameter n is called the index of refraction for a given medium and it is defined as

$$n = \frac{c}{v},$$

where c is the speed of light in vacuum and v is the speed of light in the medium. Light slows down when it goes through media. Below are some indexes of refraction.

Table 1. Media and Index of Refraction.

Medium	Index of Refraction
Vacuum	1
Air	1
Water	1.33
Glass (Typical Crown Glass)	1.5
Diamond	2.4

As we have mentioned, the relation $n_1 \sin\theta_1 = n_2 \sin\theta_2$ is today known as Snell's Law but earlier, the Persian scientist Ibn Sahl (c. 940 – 1000) used it in his optics book in designing lenses in the 980s. Willebrord Snell (c. 1580 – 1626), also known as Willebrordus Snellius, gave a derivation in 1621, but it was not published. René Descartes (1596-1650) gave a derivation in 1637. So sometimes, the law is referred to as the Snell-Descartes law, but Ibn Sahl should be in there too.

Note that with $n_1 \sin\theta_1 = n_2 \sin\theta_2$, if the index of refraction is greater, the angle is smaller and vice versa. Therefore, going from a "fast medium" where light travels faster and then enters a "slow medium" where light travels slower, the light bends towards the normal. See Fig. 7. Some light gets reflected at the interface, which is not shown in Fig. 7.

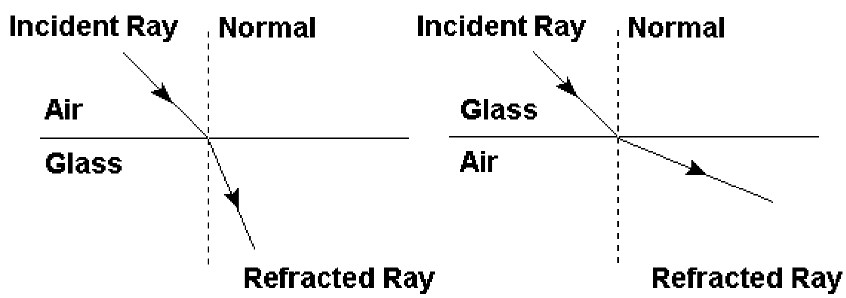

Fig. 7. Light traveling from one medium to another.

But if you go from “slow” to “fast’ you can bend away from the normal so much that you skim the interface with an angle of 90° from the normal. The incident angle in this case is called the critical angle See the three cases in Fig. 8 where the middle one illustrates the critical angle and the last case at the right is total internal reflection.

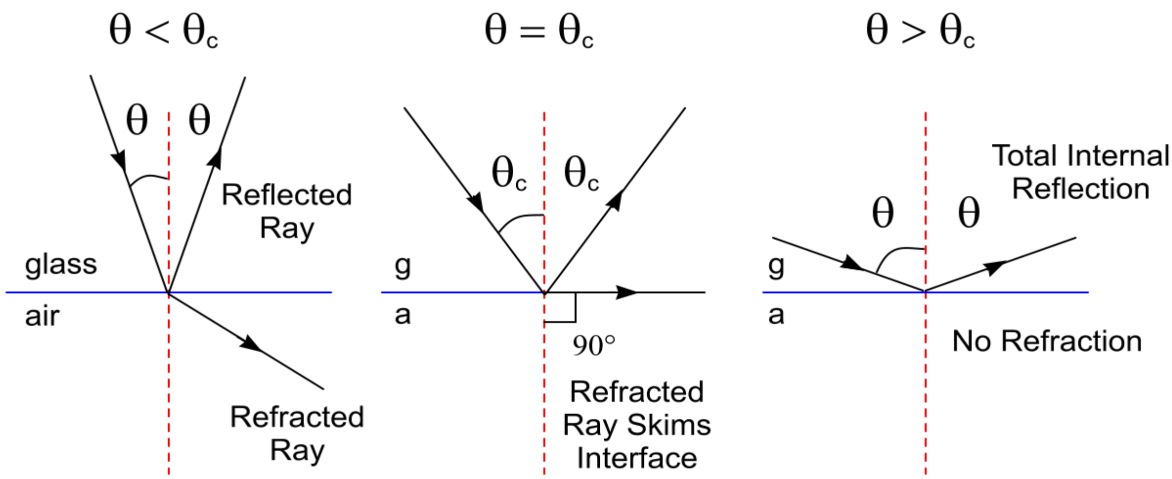

Fig. 8. The three basic cases in going from glass to air.

Let’s solve for the critical angle when light is in water and air is the surrounding medium. Snell’s Law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

becomes

$$n_w \sin\theta_c = n_a \sin 90° \quad \text{or} \quad n_w \sin\theta_c = n_a (1),$$

and we find that the critical angle is

$$\theta_c = \sin^{-1}\frac{n_a}{n_w} = \sin^{-1}\frac{1}{1.33} = 49°$$

Fig. 9. Total Internal Reflection, PHYS 101 Light and Visual Phenomena, Spring 2020
See the short video at: https://youtu.be/8yGmql7pfr4

For incident angles greater the critical angle, the light undergoes total internal reflection inside the water, as illustrated in Fig. 9. You can see a short video here: https://youtu.be/8yGmql7pfr4

Fig. 10 below includes two sketches of total internal reflection, one for fiber optics and one for the water jet stream.

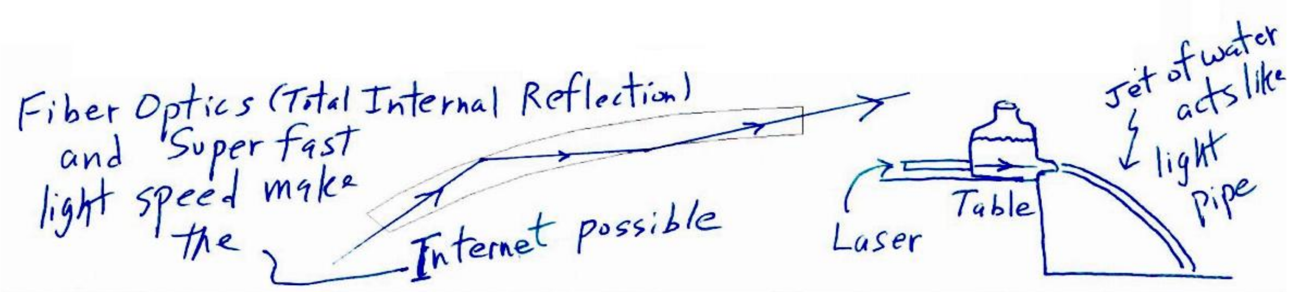

Fig. 10. Total Internal Reflection in fiber optics and a jet of water.

Total internal reflection makes fiber optics possible and the Internet! The fast speed of light also makes the Internet possible. Think about how physics here revolutionized the world with the Internet. The two main ingredients are optical:

1. Fiber optics and Total Internal Reflection.
2. The Super Fast Speed of Light for quick world communication.

So we have used the wave model to arrive at Snell's Law. Then we used the rays of geometrical optics and Fermat's Principle. So we have two derivations, one from each of two of the basic realms of optics:

1. Geometrical Optics – light travels in straight lines.
2. Physical Optics – light as waves (crests and troughs).

The rows of soldiers are the crests of the wave, like a water wave. In between we can call the center of the gap between the rows as the bottoms of the troughs. Note that in physical optics you have a direction of travel – a ray! But in geometrical optics for our derivation using the principle of least time, there are no crests or waves. So you see, the wave model contains the ray concept. The directional ray is perpendicular to the crests.

However, the Principle of Least Time allows the ray model of geometrical optics to make a connection to wave optics via its derivation of Snell's Law. The Principle of Least Time is a very sophisticated approach to refraction.

B4. Mirages

Fig. 12. Road surface mirage. It appears that water is present.

The common road surface water-mirage seen in Fig. 12. It is due to refraction that occurs with a gradual change in index of refraction. The index of refraction changes since the layers of air have different temperatures. The index of refraction is less for air at hotter temperatures. Therefore, light going from a layer of cold to warmer air bends away from the normal as seen in Fig. 13. Total internal reflection is shown at the bottom of the path, which is incorrect. The ray diagram of geometrical optics is too limited here.

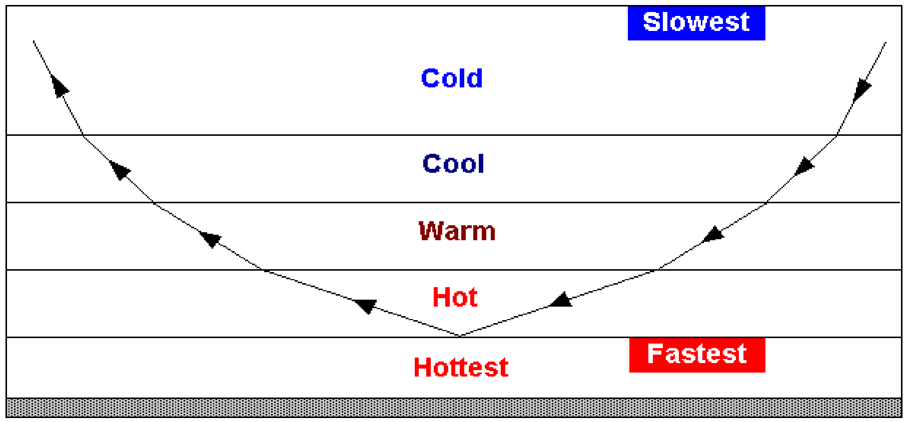

Fig. 13. Simplified mirage model with incorrect total internal reflection at bottom.

If the ray at the bottom is replaced with a wave crest, the bottom of the wave crest will travel faster than the top part of the crest, making the wave path bend upward. The observer at the left in Fig. 14 perceives an inverted image, which the brain processes as coming from a reflection of water midway between the observer and the tree. The water is the mirage. Since the image of the tree is locate underneath the real tree, the mirage is classified as an inferior mirage.

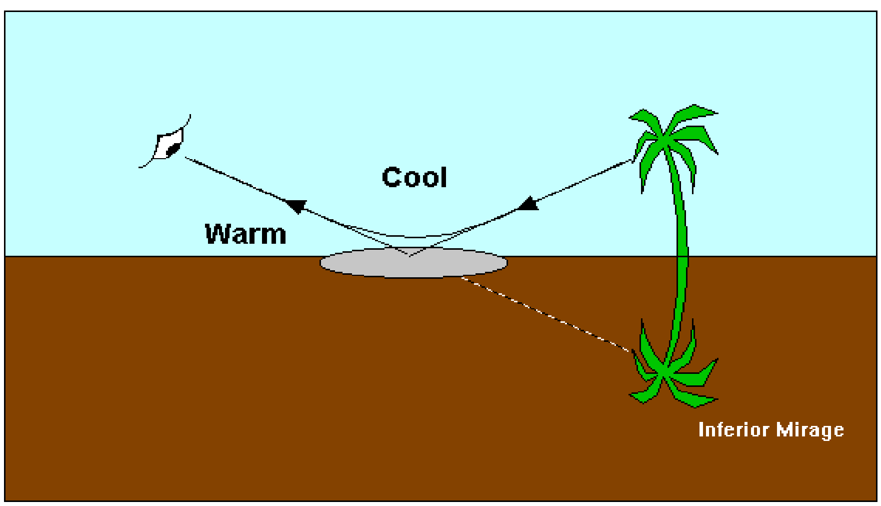

Fig. 14. Inferior mirage caused by refraction.

A simple model of three levels of air with different temperatures is shown in Fig. 15. The air temperature gets higher as you near the hot road surface below. Note that the first angle and the last one are related directly without the middle layer in the formula, i.e., $n_1 \sin\theta_1 = n_3 \sin\theta_3$.

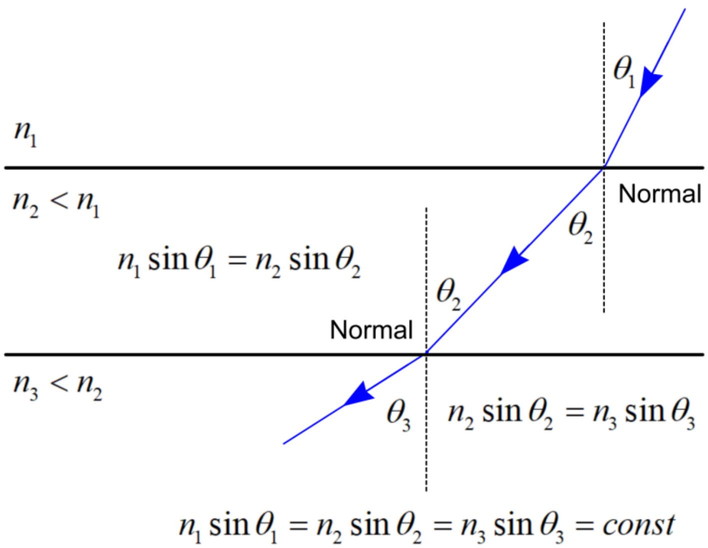

Fig. 15. Model with three layers of air, air getting warmer towards the ground.

So we only need the cold index of refraction at the upper level and the hot index of refraction near the road surface. Note that the angle of incidence near the hot road surface is a 90° angle as shown in Fig. 16. The relation $n_1 \sin\theta_1 = n_3 \sin\theta_3$ with $n_1 = n_{\text{cold}}$ at the top layer and $n_3 = n_{\text{hot}}$ at the lowest level leads to $n_{\text{cold}} \sin\theta = n_{\text{hot}} \sin 90° = n_{\text{hot}}$.

In Fig. 16 let the incident angle at the observer's height at the far left, where $n = n_{\text{cold}}$, be θ . Then remember that $n_{\text{cold}} \sin\theta$ is a constant and will be equal to $n_{\text{hot}} \sin 90° = n_{\text{hot}}$.

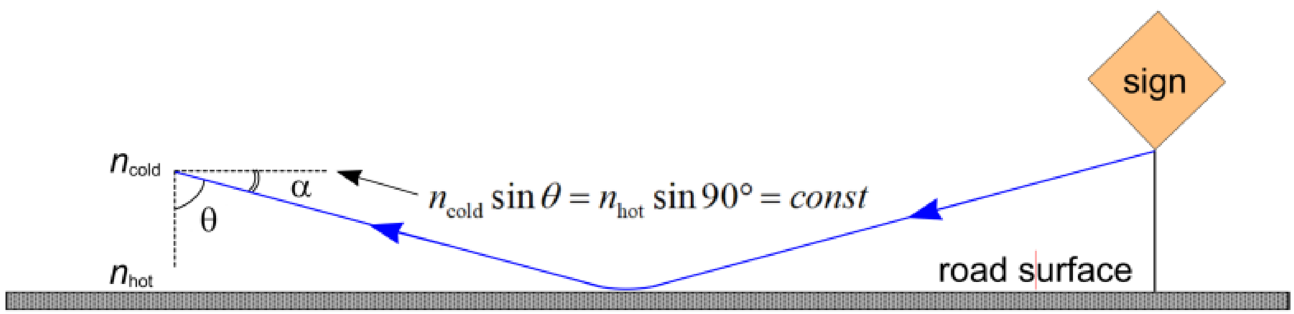

Fig. 16. Refracted light path. The most change occurs near the hot road.

The angle α in Fig. 16 is called the mirage angle. Note that $\theta + \alpha = 90°$. Therefore, substituting $\theta = 90° - \alpha$ in $n_{\text{cold}} \sin\theta = n_{\text{hot}}$ leads to

$$n_{\text{cold}} \sin(90° - \alpha) = n_{\text{hot}} \quad \text{and} \quad n_{\text{cold}} \cos\alpha = n_{\text{hot}} \, .$$

The simple result is $\cos\alpha = \dfrac{n_{\text{hot}}}{n_{\text{cold}}}$. Since the mirage angle will be very small, we can expand the cosine:

$$1 - \frac{\alpha^2}{2} = \frac{n_{\text{hot}}}{n_{\text{cold}}},$$

leading to $\dfrac{\alpha^2}{2} = 1 - \dfrac{n_{\text{hot}}}{n_{\text{cold}}}$ and

$$\alpha = \sqrt{2\left(1 - \frac{n_{\text{hot}}}{n_{\text{cold}}}\right)}$$

We would like to have a formula that brings in temperature. Two British researchers, a chemist and a priest, made a fascinating discovery about index of refraction. The chemist John Hall Gladstone and the Reverend Thomas Pelham Dale presented a paper in 1863 to the Royal Society of London on their empirical studies of index of refraction and density. They found that to a good approximation, the increase in index of refraction from n = 1 is proportional to the density of the medium. Their observation is known as the Gladstone-Dale law and is expressed in terms of the density ρ as

$$n - 1 = \text{const} \cdot \rho.$$

But the density of air can be related to the temperature through the ideal gas law

$$PV = nRT.$$

From the cool air height to the hot road surface, the atmospheric pressure P is constant. Therefore,

$$P = \frac{nRT}{V} = \text{const}.$$

The density of the air ρ is proportional to $\dfrac{n}{V} = \dfrac{P}{RT} = \dfrac{const}{T}$ since atmopsheric weather pressure will be constant over our region near the ground. Therefore $\rho = \dfrac{\text{const}}{T}$.

Combining $n-1=\text{const}\cdot\rho$ and $\rho=\dfrac{\text{const}}{T}$ gives

$$n-1=\frac{\text{const}}{T} \quad \text{and} \quad (n-1)T=\text{const}.$$

Returning to $\alpha=\sqrt{2(1-\dfrac{n_{\text{hot}}}{n_{\text{cold}}})}$, we can incorporate $(n-1)T=\text{const}$. First note that

$$(n_{\text{cold}}-1)T_{\text{cold}}=(n_{\text{hot}}-1)T_{\text{hot}}=\text{const}.$$

Now solve for $1-\dfrac{n_{\text{hot}}}{n_{\text{cold}}}$. The steps are below, where we abbreviate hot as h and cold as c.

$$(n_{\text{c}}-1)T_{\text{c}}=(n_{\text{h}}-1)T_{\text{h}}$$

$$n_{\text{c}}T_{\text{c}}-T_{\text{c}}=n_{\text{h}}T_{\text{h}}-T_{\text{h}}$$

$$T_{\text{c}}-\frac{1}{n_{\text{c}}}T_{\text{c}}=\frac{n_{\text{h}}}{n_{\text{c}}}T_{\text{h}}-\frac{1}{n_{\text{c}}}T_{\text{h}}$$

$$\frac{n_{\text{h}}}{n_{\text{c}}}T_{\text{h}}=T_{\text{c}}-\frac{1}{n_{\text{c}}}T_{\text{c}}+\frac{1}{n_{\text{c}}}T_{\text{h}}$$

$$\frac{n_{\text{h}}}{n_{\text{c}}}=\frac{T_{\text{c}}}{T_{\text{h}}}-\frac{1}{n_{\text{c}}}\frac{T_{\text{c}}}{T_{\text{h}}}+\frac{1}{n_{\text{c}}}$$

$$\frac{n_{\text{h}}}{n_{\text{c}}}=\frac{T_{\text{c}}}{T_{\text{h}}}+\frac{1}{n_c}\left[1-\frac{T_{\text{c}}}{T_{\text{h}}}\right]$$

$$1-\frac{n_\mathrm{h}}{n_\mathrm{c}} = 1-\frac{T_\mathrm{c}}{T_\mathrm{h}} - \frac{1}{n_c}\left[1-\frac{T_\mathrm{c}}{T_\mathrm{h}}\right]$$

$$1-\frac{n_\mathrm{h}}{n_\mathrm{c}} = \left[1-\frac{1}{n_c}\right]\left[1-\frac{T_\mathrm{c}}{T_\mathrm{h}}\right].$$

Finally,

$$\alpha = \sqrt{2(1-\frac{n_\mathrm{h}}{n_\mathrm{c}})} = \sqrt{2\left[1-\frac{1}{n_c}\right]\left[1-\frac{T_\mathrm{c}}{T_\mathrm{h}}\right]}.$$

Typical ambient temperature on a hot summer day in North Carolina is

T_c = 88 °F = 31 °C = 304 K.

In contrast, the hot asphalt can be

T_h = 120 °F = 49 °C = 322 K.

To get an estimate of the angle α , we can use for the index of refraction of air at T_c to be

n_c = 1.00026.

Then,

$$\alpha = \sqrt{2\left[1-\frac{1}{n_c}\right]\left[1-\frac{T_\mathrm{c}}{T_\mathrm{h}}\right]} = \sqrt{2\left[1-\frac{1}{1.00026}\right]\left[1-\frac{304}{322}\right]} = 5.39\cdot 10^{-3}\,\mathrm{rad}$$

$$\alpha = 5.39\cdot 10^{-3}\,\mathrm{rad}\cdot\frac{180°}{\pi\ \mathrm{rad}} = 0.31°$$, a very small angle.

You will be given a homework assignment where you investigate data I published recently applying these ideas driving on hot summer roads in North Carolina.

Michael J. Ruiz, “Road Mirage Angle,” Physics Education 54, 065009 (November 2019). **pdf** and **Video Abstract**