C1. Introduction to Rainbows

Fig. 1. UNCA Rainbow, 7:45 pm, August 4. 2003. Photographer Mike Honeycutt (IT).

A rainbow visible from our school, UNCA, is seen in Fig. 1. The photographer was Mike Honeycutt form UNCA Instructional Technology (IT). Mike was at UNCA in the evening as he was often working during off hours to keep the UNCA computers functioning on campus. He told me that when he saw the rainbow, he thought of taking some photos for my *PHYS 101 Light and Visual Phenomena*, a course he took when he was a student at UNCA. He retired around 2010.

The brighter rainbow is called the primary rainbow, while the dimmer one is the secondary rainbow. The rainbow angle is the angle measured by the ground observer from the horizontal to the rainbow's apex. We will do the calculation for the primary rainbow. If you look at the top of the primary rainbow behind the tree, you cam see the angle from the horizontal to the top is about 45°. If you imagine standing on the sidewalk and lifting your arm to point to the top of the primary rainbow, you arm is extended about halfway to the sky right above you, the zenith. Since from horizontal to zenith is 90°, we can estimate the rainbow angle to be 45°. It is always a powerful goal to estimate a research answer before you do a detailed study. Of course, you might not always be able to arrive at such an estimate. But you should always try.

Fig. 2. Cup of Coffee at the End of a Rainbow. Courtesy Commander John Bortniak, NOAA.

But first check out Fig. 2 showing us what is found at the end a rainbow. The photo was taken in September 1992 by Commander John Bortniak, NOAA Corps. NOAA stands for National Oceanic and Atmospheric Administration, in the U.S. Department of Commerce.

Note that red is on top for a primary rainbow. For the secondary rainbow, the colors are reversed with the blue end of the spectrum on top. Compare the colors for the two rainbows seen in Fig. 1.

To help visualize why red is on top, see Fig. 3 below. White light enters the water drop. The refracted rays bend towards the normal. The difference between the blue and red rays is highly exaggerated for easy visualization. The blue ray goes over the observer's head in the left figure. Therefore, you see red on top. The other colors come from drops that are lower.

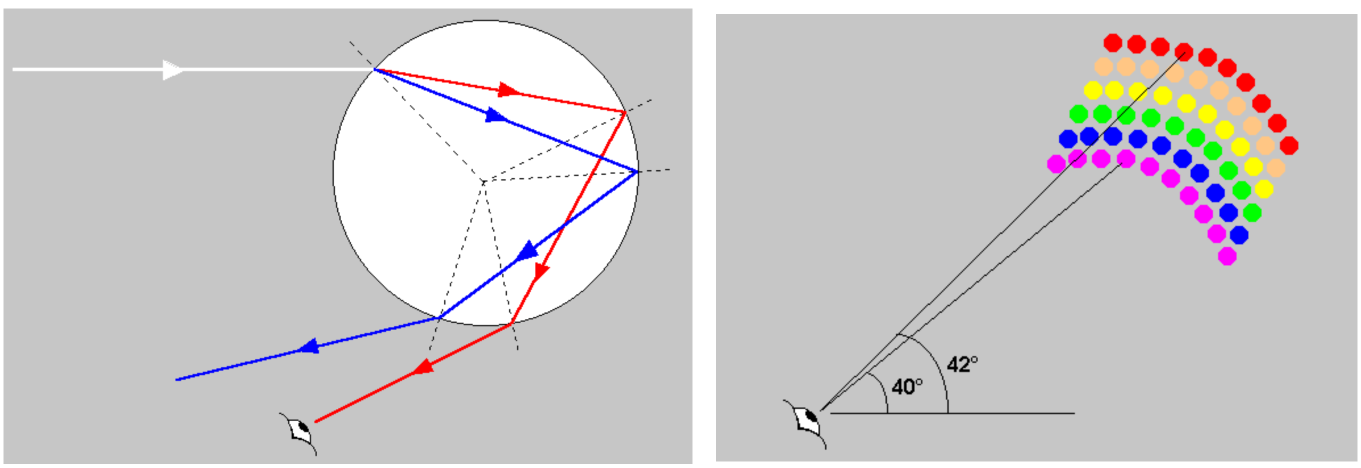


Fig. 3. Sketch of Rays for Primary Rainbow. Red is on Top.

Note the reflection at the back of the drop. The incident and reflected angles are equal. We ignore the portion of light that goes out the back of the drop. We also ignore reflected rays as the white light enters the drop and the blue and red rays leave the drop. Otherwise, the figure would be too cluttered. Our goal in this chapter is to drive the 40° rainbow angle for the primary rainbow. See Fig. 4 for the order of the colors in a primary rainbow. The order from top to bottom is Red, Orange, Yellow, Green, Blue, and Violet. The classic memory trick is ROY G. BIV, where R = Red, O = Orange, Y = Yellow, G = Green, B = Blue, I = Indigo, and V = Violet.

Fig. 4. Primary Rainbow over the Yellowstone Landscape, Wyoming
Courtesy Photographer Todd Cravens, Photo via [Good Free Photos]

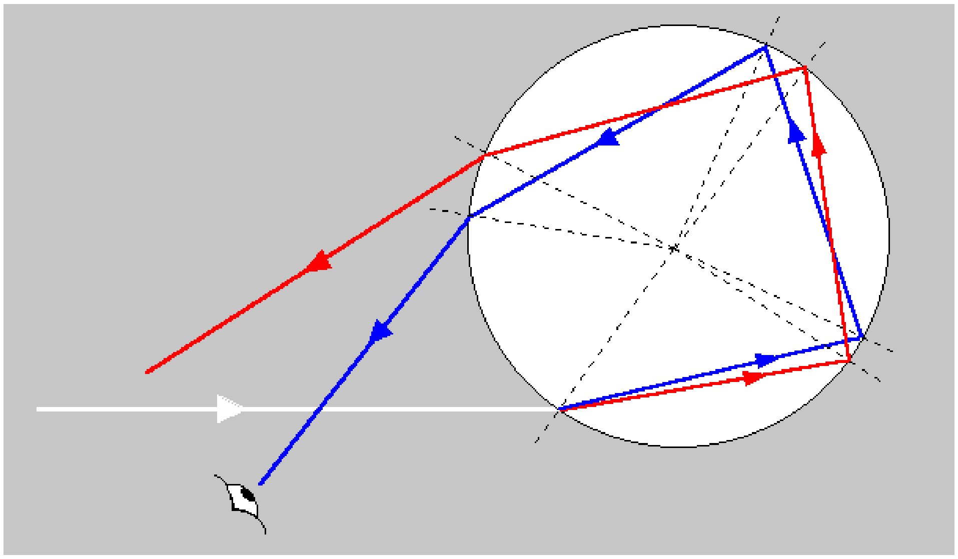


Fig. 5. Sketch of Rays for Secondary Rainbow. Blue (Violet) is on Top.

Fig. 5 illustrates the secondary rainbow. There are two reflections inside the drop and the outgoing light is therefore dimmer. Note that the white light enters the drop here in the bottom half of the raindrop when compared to the case of the primary rainbow of Fig. 3. The blue end of the spectrum is on top for the secondary rainbow and the spectrum is reversed.

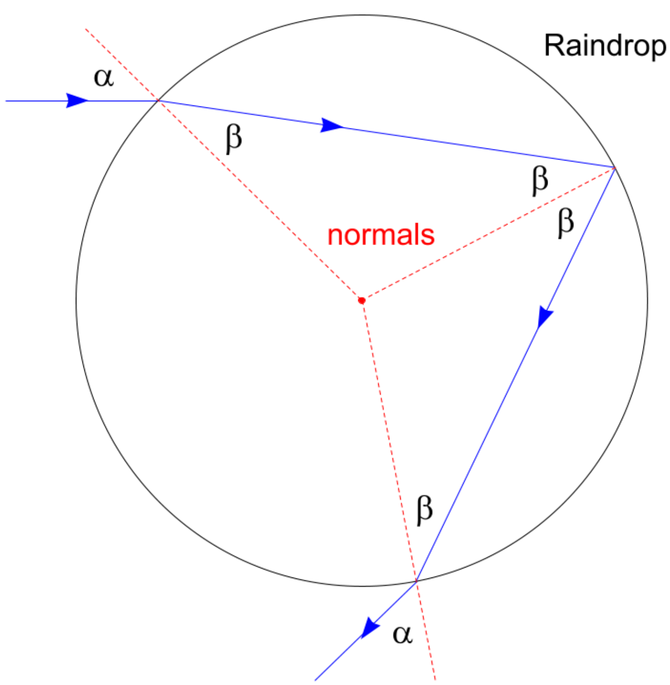


Fig. 6a. Raindrop.

Light enters the raindrop of Fig. 6a with an angle of incidence α, the angle measured from the ray to the normal. The normal's are the red dashed lines. They are perpendicular to the circumference in all cases since they emanate from the center of the circle.

The angle β is the refracted angle.

Note that when β is an incident angle for reflection that the reflected angle is also β. Four angles are marked β since they are in two isosceles triangles. The equal sides of these triangles are radii. To understand that final α. Reverse the ray direction and you have incident α coordinated with refracted β similar to the α and β at the upper left of the drop.

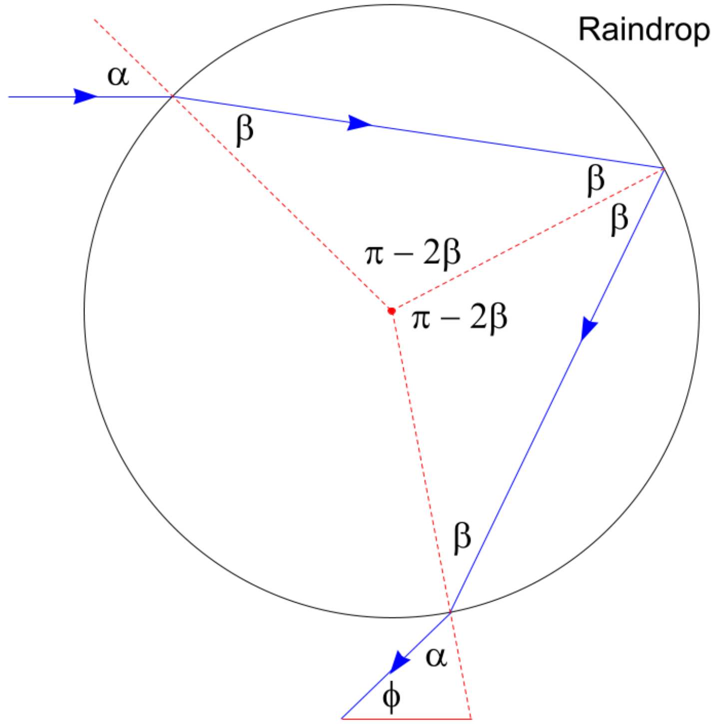


Fig. 6b. Raindrop.

In Fig. 6b the rainbow angle ϕ is introduced. This angle can be thought of as an elevation angle. If you take your hand at first extended in front of you and then raise it to point to the rainbow, then you have the rainbow angle.

At this stage we do not consider that the colors spread out. Colors spread out since the index of refraction slightly varies for different wavelengths. The spread of colors due to variation in index of refraction is called dispersion.

Note also in Fig. 6b that we have identified the third angle in the two isosceles triangles.

A triangle has 180°. i.e., π radians. Therefore, the third angle in an isosceles triangle with two β angles is $\pi - 2\beta$.

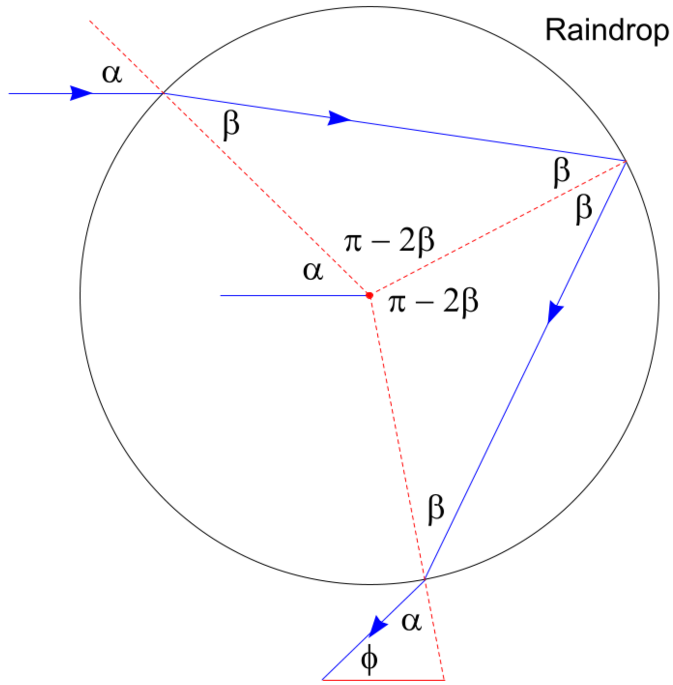


Fig. 6c. Raindrop.

We add a short horizontal line left of center in Fig. 6c. Note that this horizontal line is parallel to the horizontal incoming light ray that hits the raindrop at the upper left.

Therefore, the angle made from the horizontal to the common dashed red line must be the same angle α .

Most sources and texts will give one master figure for the raindrop, which requires you to stare at it to figure things out. Here, we build up the raindrop figure gradually so that you can see each step of the geometry unfold.

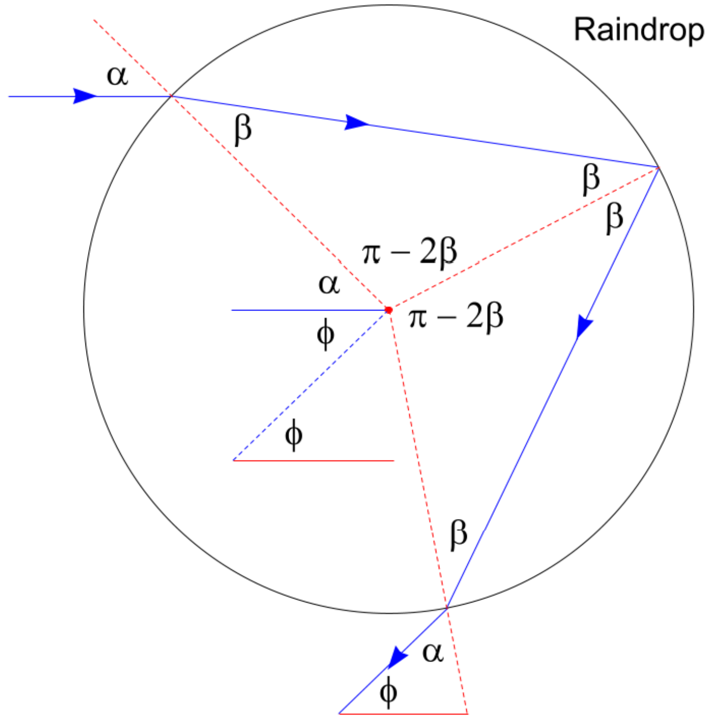


Fig. 6d. Raindrop.

We add a slanted dashed blue line parallel to the solid blue exit light ray and a short horizontal red line.

The angle between the slanted dashed blue line and the horizontal red line is ϕ as it is a reproduction of the blue-red line pair at the bottom of the figure. The angle ϕ is also the angle between the horizontal blue line and slanted dashed blue line due to alternating interior angles.

I have always nicknamed the alternating interior angles rule from high school geometry as the "Z" or "Zorro" rule since as a kid I watched Zorro making Z letters with his sword (Zorro, 1957 Disney TV Series). Can you find another α angle somewhere in Fig. 6d? See below for the answer.

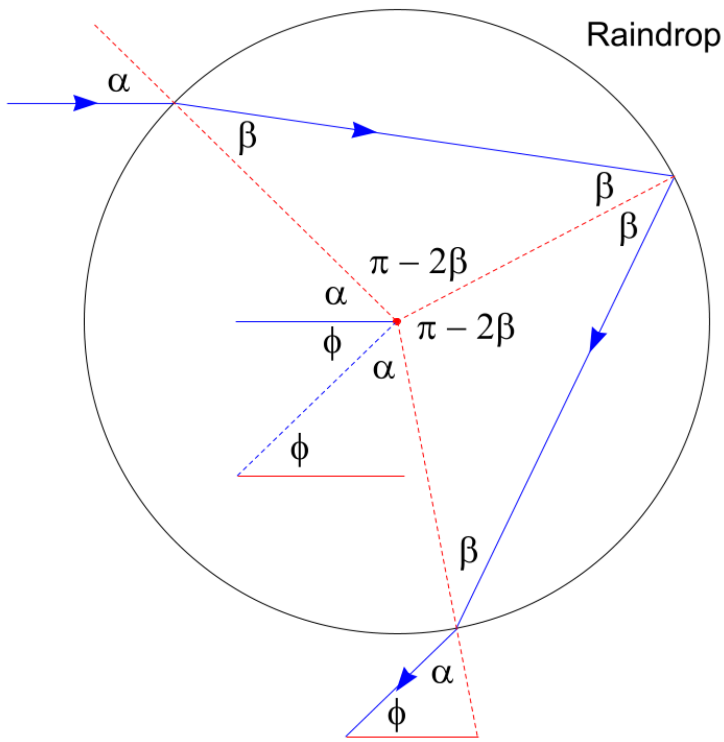


Fig. 6e. Raindrop.

The α angle added in Fig. 6e works because the diagonal blue dashed line is parallel to the exit light ray blue line and these slanted lines meet with the same dashed red line. We are now all set. The basic geometry has been established. The next step is to set all the angles in the middle of the circle equal to $360° = 2\pi$ radians.

$$\alpha + 2(\pi - 2\beta) + \alpha + \phi = 2\pi$$

$$\alpha + 2\pi - 4\beta + \alpha + \phi = 2\pi$$

$$2\alpha - 4\beta + \phi = 0$$

The final result solving for the rainbow angle

$$\phi = -2\alpha + 4\beta .$$

How do we arrive at a numerical value for this angle?

Doesn't the angle depend on where the incoming light hits the raindrop?

Doesn't the light hit the entire drop anyway?

Why is it there a special rainbow angle in the first place?

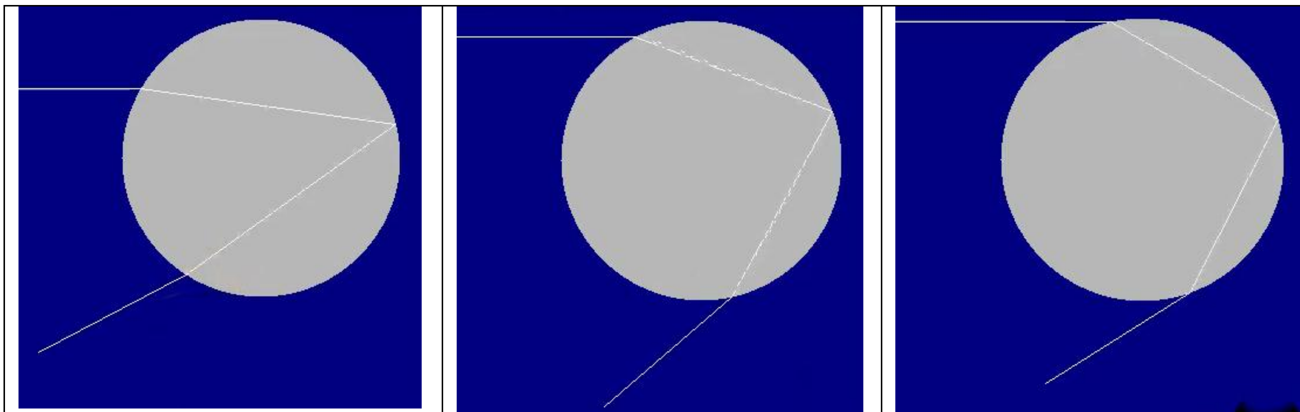
As I was writing this page, the very talented artist photographer Colin Leonhardt sent me an email from Australia giving me permission to use his fantastic rainbow photo below, which illustrates the rainbow angle for a complete rainbow.

Fig. 7. A Full Circle Rainbow over Australia. Astronomy Picture of the Day 9/30/2014.
Image Credit & Copyright: Colin Leonhardt, Birdseye View Photography, Facebook Page

Question: Why is the Sun behind the photographer? From our earlier figures we see there is a reflection at the back of the raindrop. The light then comes back. So the sunlight has to be behind us as we look forward towards the rainbow. Can you tell the time of day for the UNCA rainbow in Fig. 1, knowing the campus and where the Sun had to have been?

C2. The Rainbow Angle – a Max-Min Calculus Problem

The secret to the answer of the questions posed in the previous example is illustrated in Fig. 8. Consider a single light ray entering the raindrop along the horizontal. In Fig. 8a the outgoing ray has a modest angle or steepness. In Fig. 8b the incoming ray hits the drop higher and has the steepest outgoing angle. In Fig. 8c the incoming horizontal ray hits the droplet very far up and the result is again, a modest steepness for the outgoing ray.



a. Incoming ray near middle of drop giving medium rainbow outgoing angle.	b. Incoming ray hits higher on drop giving large (steep) rainbow angle.	c. Incoming ray hits too high and the rainbow angle is not as steep.

Fig. 8. The middle gives the rainbow angle, where the outgoing angle is maximized.

Near the maximum steepness, there will be a nearby bundle of rays on either side of the maximum that serve to amplify the light. This enhanced light gives us the rainbow. So we need to find the maximum

$$\phi = -2\alpha + 4\beta ,$$

as we scan the different incoming α angles that depend on where the light ray enters the raindrop. We have a max-min problem in calculus. We need to find the α where

$$\frac{d\phi}{d\alpha} = 0 ,$$

and then plug that α into $\phi = -2\alpha + 4\beta$, which will give $\phi = \phi_{\max}$, the rainbow angle. And we expect it to be near 45°, thereabouts. But, we need to find $\beta = \beta(\alpha)$ so that we have $\phi = \phi(\alpha)$, i.e., in terms of one variable α. Then we can proceed with setting

$$\frac{d\phi(\alpha)}{d\alpha} = 0 .$$

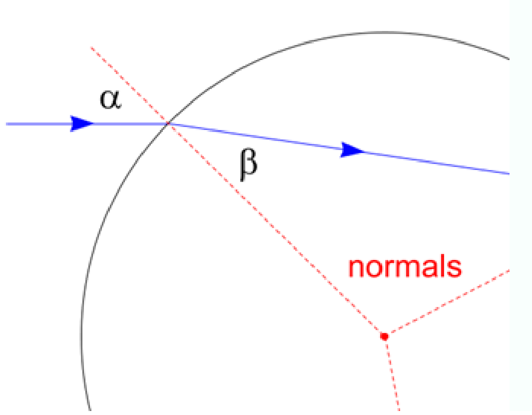


Fig. 9. Incoming-Ray Region of Fig. 6.

We can relate α and β by Snell's Law.

$$n_{\text{air}} \sin\alpha = n_{\text{water}} \sin\beta .$$

From our last chapter we know that the index of refraction for air is essentially 1. Therefore we take $n_{\text{air}} = 1$ and set $n_{\text{water}} = n$. The connection between α and β is

$$\sin\alpha = n \sin\beta ,$$

$$\beta = \sin^{-1}\left[\frac{\sin\alpha}{n}\right].$$

Then, $\phi = -2\alpha + 4\beta$ becomes

$$\phi = -2\alpha + 4\sin^{-1}\left[\frac{\sin\alpha}{n}\right].$$

In taking $\frac{d\phi(\alpha)}{d\alpha}$, we will need the derivative of the arcsine. Being a theoretical physicist, with Richard Feynman as my hero, I include derivations for all steps. Let

$$y = \sin^{-1} x .$$

Then

$$\sin y = x \quad \text{and} \quad \frac{d \sin y}{dx} = 1 .$$

But by the chain rule

$$\frac{d\sin y}{dx}=1 \quad \text{can be written as} \quad \cos y\frac{dy}{dx}=1.$$

The expression $\cos y\dfrac{dy}{dx}=1$ leads to

$$\frac{dy}{dx}=\frac{1}{\cos y}.$$

To finish, note that $\cos y=\sqrt{1-\sin^2 y}$. But, $\sin y = x$. Therefore,

$$\cos y=\sqrt{1-x^2}.$$

Putting it all together,

$$\frac{d\sin^{-1}x}{dx}=\frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-x^2}}$$, with the compact result

$$\frac{d\sin^{-1}x}{dx}=\frac{1}{\sqrt{1-x^2}}.$$

Now we are ready to calculate $\dfrac{d\phi(\alpha)}{d\alpha}$, where $\phi=-2\alpha+4\sin^{-1}\left[\dfrac{\sin\alpha}{n}\right]$ and then set the derivative equal to zero. The derivative is

$$\frac{d\phi(\alpha)}{d\alpha}=-2+4\frac{d}{d\alpha}\sin^{-1}\left[\frac{\sin\alpha}{n}\right].$$

Let $u=\dfrac{\sin\alpha}{n}$. Then

$$\frac{d}{d\alpha}\sin^{-1}\left[\frac{\sin\alpha}{n}\right]=\frac{d}{du}(\sin^{-1}u)\frac{du}{d\alpha}=\frac{1}{\sqrt{1-u^2}}\frac{du}{d\alpha}=\frac{1}{\sqrt{1-u^2}}\frac{\cos\alpha}{n}.$$

$$\frac{d}{d\alpha}\sin^{-1}\left[\frac{\sin\alpha}{n}\right]=\frac{1}{\sqrt{1-u^2}}\frac{\cos\alpha}{n}=\frac{1}{\sqrt{1-\left[\frac{\sin\alpha}{n}\right]^2}}\frac{\cos\alpha}{n}$$

Setting $\frac{d\phi(\alpha)}{d\alpha}=0$ leads to $\frac{d\phi(\alpha)}{d\alpha}=-2+4\frac{d}{d\alpha}\sin^{-1}\left[\frac{\sin\alpha}{n}\right]=0$ and

$$\frac{d\phi(\alpha)}{d\alpha}=-2+4\frac{1}{\sqrt{1-\left[\frac{\sin\alpha}{n}\right]^2}}\frac{\cos\alpha}{n}=0 .$$

Bringing the n into the square root,

$$-2+\frac{4\cos\alpha}{\sqrt{n^2-\sin^2\alpha}}=0$$

and things look simpler. We proceed to solve for the α that gives the maximum ϕ.

$$\frac{4\cos\alpha}{\sqrt{n^2-\sin^2\alpha}}=2$$

$$4\cos\alpha=2\sqrt{n^2-\sin^2\alpha}$$

$$2\cos\alpha=\sqrt{n^2-\sin^2\alpha}$$

$$4\cos^2\alpha=n^2-\sin^2\alpha$$

$$4\cos^2\alpha+\sin^2\alpha=n^2$$

$$3\cos^2\alpha+\cos^2\alpha+\sin^2\alpha=n^2$$

$$3\cos^2\alpha+1=n^2$$

$$3\cos^2\alpha=n^2-1$$

$$\cos^2 \alpha = \frac{n^2 - 1}{3}$$

When I was in high school I would always remember the 1.33 index of refraction for water as 4/3 and the 1.5 index of refraction for glass as 3/2. Using $n = 4/3$,

$$\cos^2 \alpha = \frac{n^2 - 1}{3} = \frac{1}{3}\left[(\frac{4}{3})^2 - 1 \right] = \frac{1}{3}\left[\frac{16}{9} - \frac{9}{9} \right] = \frac{1}{3}\frac{7}{9} = \frac{7}{27}.$$

Then, $\cos \alpha = \sqrt{\frac{7}{27}}$ and $\alpha = \cos^{-1}\sqrt{\frac{7}{27}} = 59.4°$.

For β we need to figure out $\beta = \sin^{-1}\left[\frac{\sin \alpha}{n} \right]$.

Since I love fractions,

$$\sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \frac{7}{27}} = \sqrt{\frac{27}{27} - \frac{7}{27}} = \sqrt{\frac{20}{27}}$$

and

$$\frac{\sin \alpha}{n} = \frac{1}{n}\sqrt{\frac{20}{27}} = \frac{3}{4}\sqrt{\frac{20}{27}}$$, leading to

$$\beta = \sin^{-1}\left[\frac{\sin \alpha}{n} \right] = \sin^{-1}\left[\frac{3}{4}\sqrt{\frac{20}{27}} \right] = 40.2°.$$

With $\alpha = 59.4°$ and $\beta = 40.2°$,

The rainbow angle $\phi = -2\alpha + 4\beta$ is

$$\phi = -2(59.4°) + 4(40.2°) = 42°.$$

Not bad, as we estimated the rainbow angle to be about 45°.

To get the range for the colors we use the extreme values

$$n(\text{violet end}) = n(\lambda = 400 \text{ nm}) = 1.345\,,$$

$$n(\text{red end}) = n(\lambda = 700 \text{ nm}) = 1.331\,.$$

$$\cos^2 \alpha_{400} = \frac{n_{400}^2 - 1}{3} = \frac{1.345^2 - 1}{3}$$

$$\alpha_{400} = \cos^{-1}\sqrt{\frac{1.345^2 - 1}{3}} = 58.71°$$

$$\alpha_{700} = \cos^{-1}\sqrt{\frac{1.331^2 - 1}{3}} = 59.53°$$

$$\beta_{400} = \sin^{-1}\left[\frac{\sin \alpha_{400}}{n_{400}}\right] = \sin^{-1}\left[\frac{\sin 58.71°}{1.345}\right] = 39.45°$$

$$\beta_{700} = \sin^{-1}\left[\frac{\sin \alpha_{700}}{n_{700}}\right] = \sin^{-1}\left[\frac{\sin 59.53°}{1.331}\right] = 40.36°$$

$$\phi_{400} = -2\alpha_{400} + 4\beta_{400} = -2(58.71°) + 4(39.45°) = 40°$$

$$\phi_{700} = -2\alpha_{700} + 4\beta_{700} = -2(59.53°) + 4(40.36°) = 42°$$

Red is on top with rainbow angle **42°**.
Violet is on bottom at **40°**.

The secondary rainbow angle will be given as a homework assignment. The answers are:

Violet is on top with rainbow angle **54°**.
Red is on bottom at **50°**.

The secondary rainbow is outside the primary one due to the larger angles. Colors are reversed and the rainbow is dimmer due to the 2 reflections.

The setup for the secondary rainbow appears in Fig. 10.

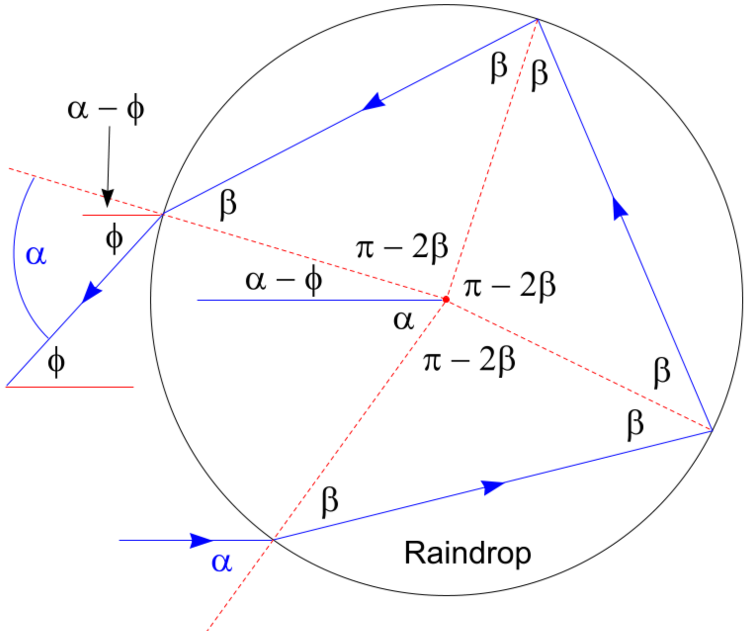


Fig. 10. Ray Diagram for Secondary Rainbow.
The incoming ray enters the lower section of the raindrop.

C3. Other Rainbow Effects

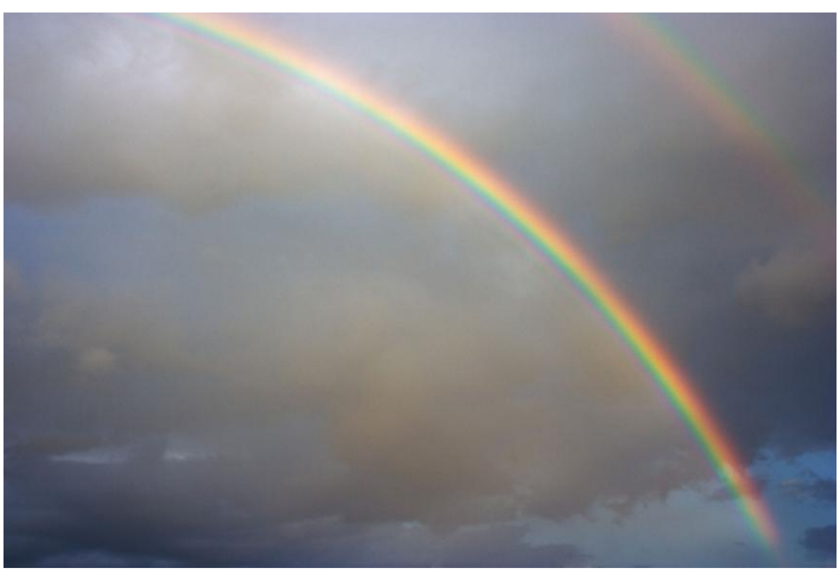


Fig. 11. Alexander's Dark Band. Courtesy Petr Kratochvil

Note the darker region between the primary and secondary rainbows. Since the light coming to our eyes from the raindrops depends on the rainbow angles, we get little light action in between the rainbows. This dark region is named Alexander's Dark Band or Alexander's Band, after Alexander of Aphrodisias who pointed it out c. 200 CE.

Fig. 12. Supernumerary Rainbow. Courtesy Mika-Pekka Markkanen

In some rainbows you might see a band of color separated from the main arc of colors at the below of the primary rainbow. See Fig. 12. These bands are due to interference effects where light taking different optical paths can reinforce at some wavelengths (constructive interference) or cancel (destructive interference) depending on their relative phases. When a specific wavelength is in phase with another wave, the crests and trough reinforce and you get color enhancement. When a crest of one wave meets the trough of a similar wave the crests and troughs cancel and you get no light.

These bands are called supernumerary bows or supernumerary rainbows. These bows cannot be explained with the light rays of geometrical optics. They occur when raindrops are small with diameters less than a millimeter are these raindrops are very close to being the same size.

Fig. 13 is the first photo of a tertiary and quaternary rainbow, taken in Germany during the summer of 2011. These rainbows are rare for two reasons. They are very dim due to the triple and quadruple reflections in the drop and they are in the general direction of the Sun.

Fig. 13. Tertiary and Quaternary Rainbows. Credit: Michael Theusner/Applied Optics