HW L1. A Trig Identity. Use Euler exponentials to show that $4\cos^3\theta = \cos(3\theta) + 3\cos\theta$.

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\cos^2\theta = \left[\frac{e^{i\theta} + e^{-i\theta}}{2}\right]^2$$

$$\cos^2\theta = \frac{e^{2i\theta} + 2 + e^{-2i\theta}}{4}$$

$$\cos^3\theta = \left[\frac{e^{i\theta} + e^{-i\theta}}{2}\right]\left[\frac{e^{2i\theta} + 2 + e^{-2i\theta}}{4}\right]$$

$$4\cos^3\theta = \frac{1}{2}\left[(e^{3i\theta} + 2e^{i\theta} + e^{-i\theta}) + (e^{i\theta} + 2e^{-i\theta} + e^{-3i\theta})\right]$$

$$4\cos^3\theta = \frac{1}{2}(e^{3i\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-3i\theta})$$

Faster to remember: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

In fact, it is always good to remember:

$$(a+b)^n = a^n b^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \ldots + a^0 b^n$$

$$4\cos^3\theta = \left(\frac{e^{3i\theta} + e^{-3i\theta}}{2} + 3\frac{e^{i\theta} + e^{-i\theta}}{2}\right) = \left[\cos(3\theta) + 3\cos\theta\right]$$

$$\boxed{4\cos^3\theta = \cos(3\theta) + 3\cos\theta}$$

HW L2. The Cauchy Formula for Dispersion.

Augustin-Louis Cauchy (1789-1857). French mathematician, physicist, and engineer. Among Cauchy's many achievements is his contributions in complex variables. In this problem we investigate his empirical formula for the index of refraction in a dispersive medium. One of his formulas is $n(\lambda) = A + \frac{B}{\lambda^2}$, where A and B are constants for the medium. A popular crown glass is made by Schott, a glass company in Mainz, Germany. The company can be traced back to a company founded in 1884 by Otto Schott, Ernst Abbe of the Abbe number, Carl Zeiss and son Roderich Zeiss. The popular universal glass borosilicate glass BK7 is used in a variety of high-quality applications.

Index of Refraction for Borosilicate Glass Schott BK7 at Three Wavelengths			
Fraunhofer Line	F	D	C
Element	Hydrogen	Sodium	Hydrogen
Description	H-beta	Doublet Average	H-alpha
Color	Blue	Yellow	Red
Wavelength	486.1 nm	589.3 nm	656.3 nm
Refractive Index n	1.522	1.517	1.514
Cauchy Result for n	**1.522**	**1.517**	**1.514**

Complete the above table to four significant figures using the Cauchy approximation where the Cauchy constants for BK7 glass are A = 1.5046 and B = 0.00420 when the wavelengths entered into the formula are in microns (μm). Include the numerical steps for each of your calculations.

$$n(\lambda) = 1.5046 + \frac{0.00420}{\lambda(\text{in } \mu\text{m})^2}$$

$$n(486.1 \text{ nm}) = 1.5046 + \frac{0.00420}{(0.4861)^2} = 1.52237 = 1.522$$

$$n(589.3 \text{ nm}) = 1.5046 + \frac{0.00420}{(0.5893)^2} = 1.51669 = 1.517$$

$$n(656.3 \text{ nm}) = 1.5046 + \frac{0.00420}{(0.6563)^2} = 1.51435 = 1.514$$

HW L3. Group Velocity. Derive the following group velocity formula in a dispersive medium.

$$v_g = \frac{c}{n}\left[1+\frac{\lambda}{n}\frac{dn}{d\lambda}\right] \text{ starting from } v_g = \frac{d\omega}{dk}.$$

There are several ways to proceed juggling the variables.

Here is one way.

Since $kv = \omega$ and $n = \frac{c}{v}$, we have $\omega = kv = \frac{kc}{n}$.

$$k = \frac{2\pi}{\lambda} \quad \Rightarrow \quad \omega = kv = \frac{2\pi}{\lambda}\frac{c}{n}$$

$$v_g = \frac{d\omega}{dk} = \frac{d}{d\lambda}\left[\frac{2\pi}{\lambda}\frac{c}{n}\right]\frac{d\lambda}{dk}$$

$$v_g = \frac{d\omega}{dk} = 2\pi c\frac{d}{d\lambda}\left[\frac{1}{\lambda n}\right]\frac{d\lambda}{dk} = 2\pi c\left[-\frac{1}{\lambda^2}\frac{1}{n} - \frac{1}{\lambda}\frac{1}{n^2}\frac{dn}{d\lambda}\right]\frac{d\lambda}{dk}$$

$$v_g = -\frac{2\pi c}{n\lambda}\left[\frac{1}{\lambda}+\frac{1}{n}\frac{dn}{d\lambda}\right]\frac{d\lambda}{dk}$$

$$\frac{d\lambda}{dk} = \frac{d}{dk}\frac{2\pi}{k} = 2\pi\frac{d}{dk}\frac{1}{k} = 2\pi\left[-\frac{1}{k^2}\right] = -\frac{2\pi}{k^2} = -2\pi\left[\frac{\lambda}{2\pi}\right]^2 = -\frac{\lambda^2}{2\pi}$$

$$v_g = -\frac{2\pi c}{n}\left[\frac{1}{\lambda^2}+\frac{1}{\lambda}\frac{1}{n}\frac{dn}{d\lambda}\right]\left[-\frac{\lambda^2}{2\pi}\right] = \frac{c}{n}\left[\frac{1}{\lambda^2}+\frac{1}{\lambda}\frac{1}{n}\frac{dn}{d\lambda}\right]\lambda^2$$

$$\boxed{v_g = \frac{c}{n}\left[1+\frac{\lambda}{n}\frac{dn}{d\lambda}\right]}$$

HW L4. Phase Velocity and Group Velocity.

a) What is the phase velocity in outer space for light emitted from the 3-2 Balmer transition, i.e., 656.3 nm, to three significant figures in km/s? This beautiful deep-red light of emission nebulae is also designated as H-alpha, Hα, or H-α.

$$v_p = \frac{\omega}{k}$$

Since $kv = \omega$ and $n = \frac{c}{v}$, we have $\omega = kv = \frac{kc}{n}$.

$$v_p = \frac{\omega}{k} = \frac{1}{k}\frac{kc}{n} = \frac{c}{n} = \frac{c}{1} = c \quad \Rightarrow \quad \boxed{v_p = 3.00 \times 10^5 \ \frac{\text{km}}{\text{s}}}$$

b) Use your group velocity formula of HW L3 and the Cauchy dispersion formula of HW L2 in order to calculate the group velocity in km/s for Hα in Borosilicate Glass Schott BK7 to three significant figures?

$$v_g = \frac{c}{n}\left[1 + \frac{\lambda}{n}\frac{dn}{d\lambda}\right] \qquad n(\lambda) = A + \frac{B}{\lambda^2}$$

A = 1.5046 and B = 0.00420 when the wavelengths are entered in microns (μm).

$$\frac{dn}{d\lambda} = \frac{d}{d\lambda}(A + \frac{B}{\lambda^2}) = -\frac{2B}{\lambda^3} \qquad v_g = \frac{c}{n}\left[1 + \frac{\lambda}{n}(-\frac{2B}{\lambda^3})\right]$$

$$v_g = \frac{c}{n}(1 - \frac{2B}{n\lambda^2})$$

$$v_g = \frac{3.00 \cdot 10^5}{1.514}\left[1 - \frac{2 \cdot (0.00420)}{1.514 \cdot (0.6563)^2}\right]\frac{\text{km}}{\text{s}}$$

$$v_g = 1.9815 \cdot 10^5 \left[1 - 0.01288\right]$$

$\boxed{v_g = 1.96 \times 10^5 \ \frac{\text{km}}{\text{s}}}$ or $\boxed{v_g = 1.95 \times 10^5 \ \frac{\text{km}}{\text{s}}}$ using $c = 2.998 \times 10^5 \ \frac{\text{km}}{\text{s}}$