Modern Optics, Prof. Ruiz, UNCA doctorphys.com
Chapter P. Why is the Sky Blue? - Solutions

HW P1. Dispersion Revisited. Show that $n^2 = 1 + \frac{n_e e^2}{\varepsilon_o m(\omega_o^2 - \omega^2)}$ for $\lambda >> \lambda_o = \frac{c}{f_o} = \frac{2\pi c}{\omega_o}$ can be put in the form $n^2(\lambda) = \alpha + \frac{\beta}{\lambda^2}$. Give the constants α and β in their simplest forms in terms of n_e, e, ε_o, m, c, and λ_o.

Solution. For $\lambda >> \lambda_o$ we have $\omega_o << \omega$. Then

$$\frac{n_e e^2}{\varepsilon_o m(\omega_o^2 - \omega^2)} = \frac{n_e e^2}{\varepsilon_o m \omega_o^2 (1 - \frac{\omega^2}{\omega_o^2})} \approx \frac{n_e e^2}{\varepsilon_o m \omega_o^2}(1 + \frac{\omega^2}{\omega_o^2})$$

$$n^2 = 1 + \frac{n_e e^2}{\varepsilon_o m(\omega_o^2 - \omega^2)} \rightarrow 1 + \frac{n_e e^2}{\varepsilon_o m \omega_o^2}(1 + \frac{\omega^2}{\omega_o^2})$$

Substitute $\frac{\omega^2}{\omega_o^2} = \frac{\lambda_o^2}{\lambda^2}$ and $\frac{1}{\omega_o} = \frac{\lambda_o}{2\pi c}$

$$n^2 = 1 + \frac{n_e e^2}{\varepsilon_o m} \frac{\lambda_o^2}{4\pi^2 c^2}(1 + \frac{\lambda_o^2}{\lambda^2})$$

$$n^2 = 1 + \frac{n_e e^2 \lambda_o^2}{4\pi^2 \varepsilon_o m c^2}(1 + \frac{\lambda_o^2}{\lambda^2}) = \alpha + \frac{\beta}{\lambda^2}$$

$$\alpha = 1 + \frac{n_e e^2 \lambda_o^2}{4\pi^2 \varepsilon_o m c^2} \qquad \beta = \frac{n_e e^2 \lambda_o^4}{4\pi^2 \varepsilon_o m c^2}$$

HW P2. Scattering. From class: $E_\theta = -\dfrac{ae\sin\theta}{4\pi\varepsilon_o c^2 r}$ and $x = -\dfrac{eE_o}{m(\omega_o^2 - \omega^2)} e^{i\omega t}$.

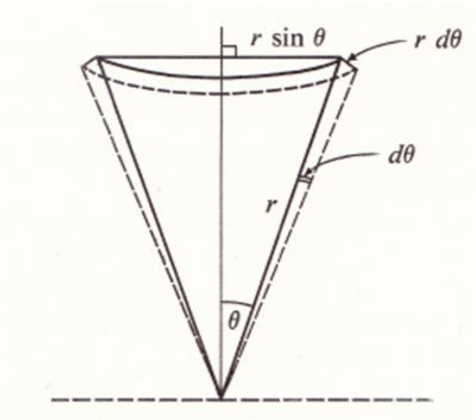
Show that the amplitude of the scattered wave is $|E_\theta| = \dfrac{e^2\omega^2 E_o \sin\theta}{4\pi\varepsilon_o c^2 r m(\omega_o^2 - \omega^2)}$.

Then show that the irradiance is $I = \dfrac{1}{2}\sqrt{\dfrac{\varepsilon_o}{\mu_o}} \dfrac{e^4\omega^4 E_o^2 \sin^2\theta}{(4\pi\varepsilon_o)^2 c^4 r^2 m^2 (\omega_o^2 - \omega^2)^2}$.

Show that for a ribbon area $dA = 2\pi r^2 \sin\theta d\theta$, the power radiated is given by

$dP = \dfrac{2\pi I_o e^4 \omega^4 \sin^3\theta}{(4\pi\varepsilon_o)^2 c^4 m^2 (\omega_o^2 - \omega^2)^2} d\theta$ where the initial incident irradiance

$I_o = \dfrac{1}{2}\sqrt{\dfrac{\varepsilon_o}{\mu_o}} E_o^2$. The power P is the reradiated power, the scattered power.



Integrate over the angle θ from $\theta = 0$ to $\theta = \pi$ and show that the total scattered power is

$$P = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\varepsilon_o mc^2}\right)^2 \frac{\omega^4}{(\omega_o^2 - \omega^2)^2} I_o .$$

Solution.

$$x = -\frac{eE_o}{m(\omega_o^2 - \omega^2)} e^{i\omega t} \quad \Rightarrow \quad a = \ddot{x} = -\frac{eE_o}{m(\omega_o^2 - \omega^2)} (-i\omega)^2 e^{i\omega t}$$

$$a = \frac{\omega^2 eE_o}{m(\omega_o^2 - \omega^2)} e^{i\omega t} \quad \Rightarrow \quad |E_\theta| = \frac{ae\sin\theta}{4\pi\varepsilon_o c^2 r} = \frac{e\sin\theta}{4\pi\varepsilon_o c^2 r} \frac{\omega^2 eE_o}{m(\omega_o^2 - \omega^2)}$$

$$I = \frac{E_o B_o}{2\mu_o} = \frac{E_o E_o}{2\mu_o c} = \frac{1}{2\mu_o}\sqrt{\varepsilon_o \mu_o} = \frac{1}{2}\sqrt{\frac{\varepsilon_o}{\mu_o}} E_o^2$$

$$I = \frac{1}{2}\sqrt{\frac{\varepsilon_o}{\mu_o}} \left[\frac{e \sin\theta}{4\pi\varepsilon_o c^2 r} \frac{\omega^2 e E_o}{m(\omega_o^2 - \omega^2)} \right]^2$$

$$I = \frac{1}{2}\sqrt{\frac{\varepsilon_o}{\mu_o}} \frac{e^4 \omega^4 E_o^2 \sin^2\theta}{(4\pi\varepsilon_o)^2 c^4 r^2 m^2 (\omega_o^2 - \omega^2)^2}$$

$$dP = I \cdot 2\pi r^2 \sin\theta d\theta$$

$$dP = \frac{1}{2}\sqrt{\frac{\varepsilon_o}{\mu_o}} \frac{2\pi E_o^2 e^4 \omega^4 \sin^3\theta}{(4\pi\varepsilon_o)^2 c^4 m^2 (\omega_o^2 - \omega^2)^2} d\theta$$

$$I_o = \frac{1}{2}\sqrt{\frac{\varepsilon_o}{\mu_o}} E_o^2$$

$$dP = \frac{2\pi I_o e^4 \omega^4 \sin^3\theta}{(4\pi\varepsilon_o)^2 c^4 m^2 (\omega_o^2 - \omega^2)^2} d\theta$$

$$P = \int_0^\pi \frac{2\pi I_o e^4 \omega^4 \sin^3\theta}{(4\pi\varepsilon_o)^2 c^4 m^2 (\omega_o^2 - \omega^2)^2} d\theta$$

$$P = \frac{2\pi e^4 \omega^4}{(4\pi\varepsilon_o)^2 m^2 c^4 (\omega_o^2 - \omega^2)^2} I_o \int_0^\pi \sin^3\theta d\theta$$

$$\int_0^\pi \sin^3\theta d\theta = \int_0^\pi \sin^2\theta \sin\theta d\theta$$

$$\int_0^{\pi} \sin^3 \theta d\theta = \int_0^{\pi} (1-\cos^2 \theta) \sin \theta d\theta$$

$$\int_0^{\pi} \sin^3 \theta d\theta = \int_0^{\pi} \sin \theta d\theta + \int_0^{\pi} \cos^2 \theta (-\sin \theta) d\theta$$

$$\int_0^{\pi} \sin^3 \theta d\theta = \int_0^{\pi} \sin \theta d\theta + \int_1^{-1} u^2 du$$

$$\int_0^{\pi} \sin^3 \theta d\theta = -\cos \theta \Big|_0^{\pi} + \frac{u^3}{3} \Bigg|_1^{-1}$$

$$\int_0^{\pi} \sin^3 \theta d\theta = -(\cos \pi - \cos 0) + \frac{1}{3} \left[(-1)^3 - (1)^3 \right]$$

$$\int_0^{\pi} \sin^3 \theta d\theta = -(-1-1) + \frac{1}{3} \left[-1-1 \right]$$

$$\int_0^{\pi} \sin^3 \theta d\theta = 2 - \frac{2}{3} = \frac{4}{3}$$

$$P = \frac{2\pi e^4 \omega^4}{(4\pi\varepsilon_o)^2 m^2 c^4 (\omega_o^2 - \omega^2)^2} I_o \int_0^{\pi} \sin^3 \theta d\theta$$

$$P = \frac{2\pi e^4 \omega^4}{(4\pi\varepsilon_o)^2 m^2 c^4 (\omega_o^2 - \omega^2)^2} I_o \frac{4}{3}$$

$$P = \frac{8\pi}{3} \left(\frac{e^2}{4\pi\varepsilon_o m c^2} \right)^2 \frac{\omega^4}{(\omega_o^2 - \omega^2)^2} I_o$$