Modern Optics, Prof. Ruiz, UNCA **doctorphys.com**
Chapter Q. The Laplacian - Solutions

HW Q1. The Electric Field.

In the second semester of introductory physics with calculus the uniform sphere of charge with total charge Q radius R is usually addressed. The charge density is

$$\rho = \frac{Q}{(4/3)\pi R^3}.$$

(a) Use Gauss's law to show that the electric field for $r > R$ is

$\vec{E}_{out}(r) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$, and that it can be written in the form $\vec{E}_{out}(r) = \frac{\rho R^3}{3\varepsilon_0 r^2}\hat{r}$.

(b) Use Gauss's law to show that the electric field for $r < R$ is $\vec{E}_{in}(r) = \frac{\rho}{3\varepsilon_0} r\,\hat{r}$.

Note that $\vec{E}_{in}(R) = \vec{E}_{out}(R) = \frac{\rho R}{3\varepsilon_0}\hat{r}$

(c) Calculate $\nabla \cdot \vec{E}$ for $r > R$.

(d) Calculate $\nabla \cdot \vec{E}$ for $r < R$.

(e) Explain your answers to (c) and (d) in light of the Maxwell equation $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$.

Note: For full credit all equations must have correct notation, e.g., vector quantities need to have arrows above them or carats for unit vectors, and a vector equation must have vector signs on each side of the equation.

Solution.

(a) $\oiint \vec{E}\cdot d\vec{A} = \frac{Q}{\varepsilon_0}$ => $E(4\pi r^2) = \frac{Q}{\varepsilon_0}$ => $E = \frac{Q}{4\pi r^2 \varepsilon_0}$

$$\boxed{\vec{E}_{out}(r) = \frac{Q}{4\pi r^2 \varepsilon_0}\hat{r}}$$

$\rho = \frac{Q}{(4/3)\pi R^3}$ => $Q = \frac{4\pi R^3}{3}\rho$ => $\vec{E} = \frac{4\pi R^3}{3}\rho\frac{1}{4\pi r^2\varepsilon_0}\hat{r}$

$$\boxed{\vec{E}_{out}(r) = \frac{\rho R^3}{3\varepsilon_0 r^2}\hat{r}}$$

(b) $\oiint \vec{E}\cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$ => $E(4\pi r^2) = \frac{1}{\varepsilon_0}\rho\frac{4}{3}\pi r^3$ => $E = \frac{1}{\varepsilon_0}\rho\frac{1}{3}r$

$$\boxed{\vec{E}_{in}(r) = \frac{\rho}{3\varepsilon_0} r\,\hat{r}}$$

(c) $\nabla\cdot\vec{E}$ for $r > R$ => $\vec{E}_{out}(r) = \frac{Q}{4\pi r^2\varepsilon_0}\hat{r}$

Spherical Coordinates

$$\nabla\cdot\vec{E} = \frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta E_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}$$

Since $E_\theta = 0$ and $E_\phi = 0$, we are left with only $E_r = \frac{Q}{4\pi r^2\varepsilon_0}$

$$\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} = \frac{1}{r^2}\frac{d}{dr}(r^2 \frac{Q}{4\pi r^2 \varepsilon_0})$$

$$\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{d}{dr}(\frac{Q}{4\pi\varepsilon_0}) = 0$$

$$\boxed{\nabla \cdot \vec{E}_{out} = 0}$$

(d) $\nabla \cdot \vec{E}$ for $r < R$ => $\vec{E}_{in}(r) = \frac{\rho}{3\varepsilon_0} r \hat{r}$

$$\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} = \frac{1}{r^2}\frac{d}{dr}(r^2 \frac{\rho}{3\varepsilon_0} r)$$

$$\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} = \frac{\rho}{3\varepsilon_0}\frac{1}{r^2}\frac{d}{dr}(r^3)$$

$$\nabla \cdot \vec{E} = \frac{\rho}{3\varepsilon_0}\frac{1}{r^2} 3r^2$$

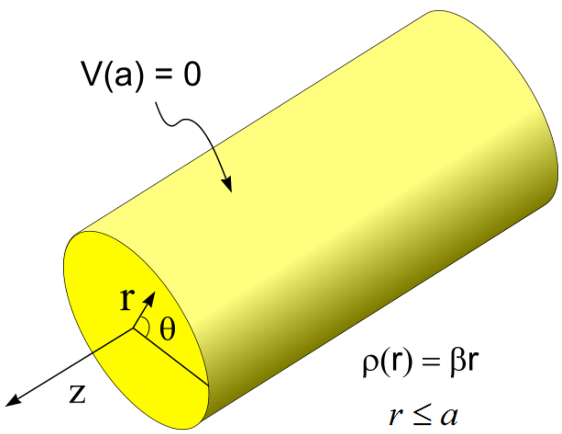
$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}$$

(e) The Maxwell equation is satisfied.

Outside is free space where there is no charge density. Therefore $\nabla \cdot \vec{E}_{out} = 0$.

Inside the sphere we have a density ρ. Therefore, $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$.

HW Q2. Poisson's Equation. Consider cylindrical coordinates defined with the notation as shown in the figure. You will see there a long cylinder which you can take to be infinite in length along the z-axis. It has radius $r = a$ and charge density $\rho(r) = \beta r$.



The following equations, one of which includes the Laplacian, describe the physics of the electric potential V and electric field **E**.

$$\vec{E} = -\nabla V$$

$$\nabla \cdot \vec{E} = \nabla \cdot (-\nabla V) = -\nabla^2 V$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad => \quad \nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

(a) Solve Poisson's equation (the one with the Laplacian) in the charge region using cylindrical coordinates to obtain the following with integration constanta A and B:

$V_{\text{in}}(r) = -\frac{\beta}{\varepsilon_0}\frac{r^3}{9} + A \ln r + B$, where "in" refers to $r \le a$. Out means $r \ge a$.

(b) Give a physics reason why the integration constant A should be taken to be zero. Then take the negative gradient of your potential to find $\vec{E}_{\text{in}}(r)$. Integrate the charge density to find the total charge Q for a section of length h of the cylinder having the full radius r = a. Then divide by h to find the linear charge density $\lambda = \frac{Q}{h}$ in terms of a and β. You can check your answer by plugging your λ into the classic electric-field formula for a line of charge: $\vec{E}_{\text{out}}(r) = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$. Your answer is most likely correct if your $\vec{E}_{\text{in}}(a)$ matches $\vec{E}_{\text{out}}(a)$.

Solution. $$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$ with $\rho(r) = \beta r$

Cylindrical coordinates with the notation given in this problem.

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 V(r) = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V(r)}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 V(r)}{\partial\theta^2} + \frac{\partial^2 V(r)}{\partial z^2}$$

$$\nabla^2 V(r) = \frac{1}{r}\frac{d}{dr}(r\frac{dV(r)}{dr}) + 0 + 0$$ since there is no θ and no z in ρ .

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} = -\frac{\beta r}{\varepsilon_0} \quad => \quad \frac{1}{r}\frac{d}{dr}(r\frac{dV}{dr}) = -\frac{\beta r}{\varepsilon_0}$$

$$\frac{d}{dr}(r\frac{dV}{dr}) = -\frac{\beta r^2}{\varepsilon_0}$$

$$(r\frac{dV}{dr}) = -\int\frac{\beta r^2}{\varepsilon_0}dr$$

$$r\frac{dV}{dr} = -\frac{\beta}{\varepsilon_0}\frac{r^3}{3} + A \quad => \quad \frac{dV}{dr} = -\frac{\beta}{\varepsilon_0}\frac{r^2}{3} + \frac{A}{r}$$

$$V = -\frac{\beta}{\varepsilon_0}\frac{r^3}{9} + A\ln r + B$$

Take $A = 0$ as $V(0)$ must not blow up to infinity.

$$V_{\text{in}}(r) = -\frac{\beta}{\varepsilon_0}\frac{r^3}{9} + B$$

$$\vec{E}_{\text{in}}(r) = -\nabla V_{\text{in}}(r)$$

$$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{k}\frac{\partial f}{\partial z}$$

$$\vec{E}_{\text{in}}(r) = -\nabla V_{\text{in}}(r) = -\frac{dV(r)}{dr}\hat{r}$$

$$\vec{E}_{\text{in}}(r) = \frac{\beta}{\varepsilon_0}\frac{r^2}{3}\hat{r}$$

Note: $E_{\text{in}}(a) = \frac{\beta}{\varepsilon_0}\frac{a^2}{3}$

$$Q_h = \int_{r=0}^{a} \rho(r) h 2\pi r dr = 2\pi\beta h \int_{r=0}^{a} r^2 dr = 2\pi\beta h \left.\frac{r^3}{3}\right|_0^a = \frac{2}{3}\beta\pi a^3 h$$

$$\lambda = \frac{Q}{h} = \frac{2}{3}\beta\pi a^3 h \cdot \frac{1}{h} = \frac{2}{3}\beta\pi a^3$$ => $$\boxed{\lambda = \frac{2\pi}{3}\beta a^3}$$

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{1}{2\pi\varepsilon_0 r}\frac{2}{3}\beta\pi a^3 = \frac{\beta}{\varepsilon_0}\frac{a^3}{3r}$$

$$E(a) = \frac{\beta}{\varepsilon_0}\frac{a^2}{3}$$