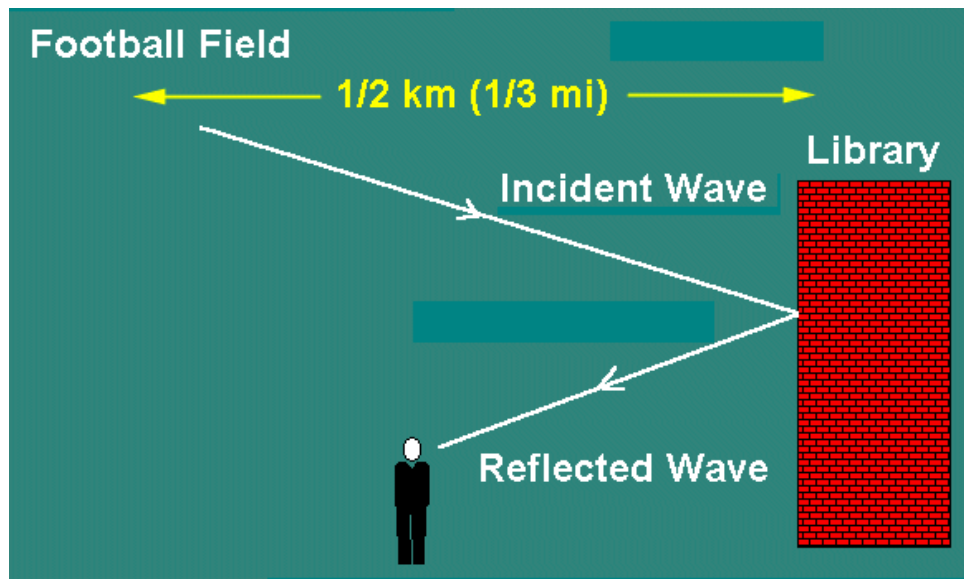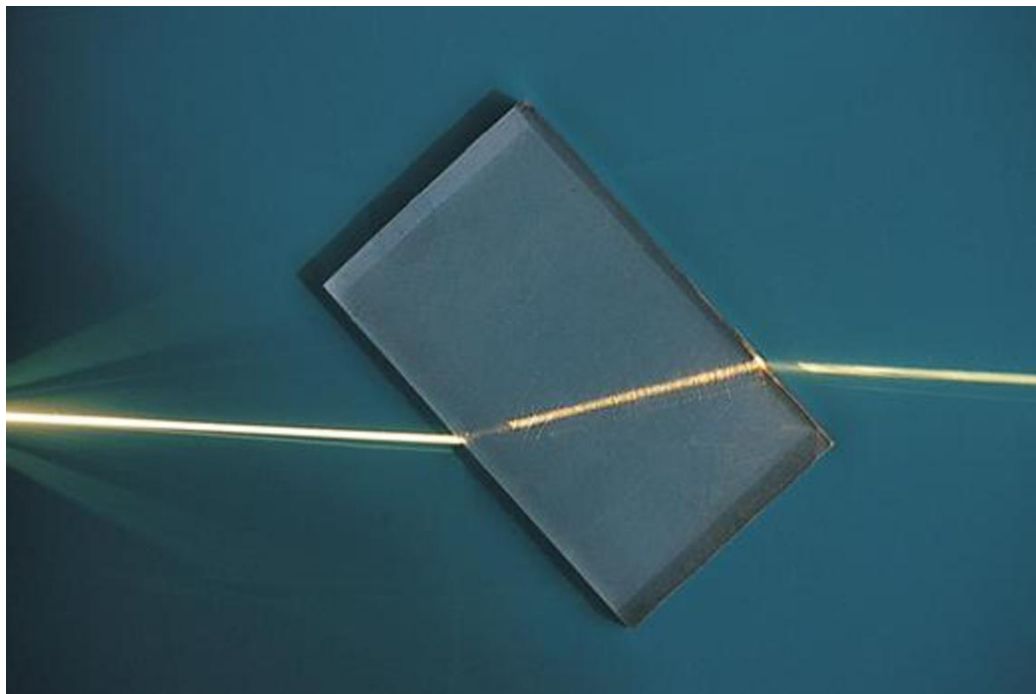**S0. Wave Properties**

1. Reflection

2. Refraction

Plastic Block. Courtesy Wikipedia. Author ajizai. Released into the Public Domain.

3. Diffraction

Courtesy The Physics Teacher, Copyright (c) Sabina Zigman
High School Student Photo, Beach in Tel Aviv, Israel

4. Interference

Courtesy Paul Doherty

S1. Spherical Waves. The wave equation in three dimension, which we have derived earlier, is

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} .$$

The Laplacian in spherical coordinates, which we have derived earlier, is

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta}(\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

We consider a spherically symmetric situation so that

$$\psi(\vec{r},t) = \psi(r,t) \text{, i.e., } \psi(r,\theta,\phi,t) = \psi(r,t) .$$

The Laplacian then reduces to the derivatives in r since $\dfrac{\partial \psi(r,t)}{\partial \theta} = \dfrac{\partial \psi(r,t)}{\partial \phi} = 0$.

$$\nabla^2 \rightarrow \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r})$$

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} \quad => \quad \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \frac{\partial \psi}{\partial r}) = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

Try $\psi(r,t) = \dfrac{R(r)}{r} e^{-i\omega t}$ since $\psi(r,t) * \psi(r,t) \sim \dfrac{1}{r^2}$.

Substituting $\psi(r,t) = \dfrac{R(r)}{r} e^{-i\omega t}$ into $\dfrac{1}{r^2} \dfrac{\partial}{\partial r}(r^2 \dfrac{\partial \psi}{\partial r}) = \dfrac{1}{c^2} \dfrac{\partial^2 \psi}{\partial t^2}$ gives

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}(\frac{R(r)}{r} e^{-i\omega t}) \right] = \frac{1}{c^2} \frac{\partial^2}{\partial t^2}(\frac{R(r)}{r} e^{-i\omega t})$$

$$e^{-i\omega t} \frac{1}{r^2} \frac{d}{dr}\left[r^2 \frac{d}{dr}(\frac{R(r)}{r}) \right] = \frac{1}{c^2} \frac{R(r)}{r} \frac{d^2(e^{-i\omega t})}{dt^2}$$

$$e^{-i\omega t} \frac{1}{r^2} \frac{d}{dr}\left[r^2 \frac{d}{dr}(\frac{R(r)}{r}) \right] = \frac{1}{c^2} \frac{R(r)}{r}(-i\omega)^2 e^{-i\omega t} = -\frac{\omega^2}{c^2} \frac{R(r)}{r} e^{-i\omega t}$$

Remember $c = \dfrac{\omega}{k}$. Therefore $\dfrac{1}{c^2} \dfrac{\partial^2 \psi}{\partial t^2} = -k^2 \dfrac{R(r)}{r} e^{-i\omega t}$.

$$\frac{1}{r^2} \frac{d}{dr}\left[r^2 \frac{d}{dr}(\frac{R}{r}) \right] = -k^2 \frac{R}{r}$$

$$\frac{1}{r^2} \frac{d}{dr}\left[r^2 (\frac{R'}{r} - \frac{R}{r^2}) \right] + k^2 \frac{R}{r} = 0$$

$$\frac{1}{r^2} \frac{d}{dr}\left[rR' - R \right] + k^2 \frac{R}{r} = 0$$

$$\frac{1}{r^2}(R' + rR'' - R') + k^2 \frac{R}{r} = 0$$

$$\frac{1}{r^2}(rR'') + k^2 \frac{R}{r} = 0$$

$$\frac{R''}{r} + k^2 \frac{R}{r} = 0$$

$$R'' + k^2 R = 0$$

$$\frac{d^2 R(r)}{dr^2} = -k^2 R(r)$$

$$R(r) = Ae^{\pm ikr} \quad => \quad \psi(r,t) = \frac{R(r)}{r} e^{-i\omega t} \quad => \quad \psi(r,t) = C \frac{e^{\pm ikr - i\omega t}}{r}$$

The wave moving outward award from some source of light is

$$\psi(r,t) = C\,\frac{e^{+ikr-i\omega t}}{r} \quad => \quad \psi(r,t) = C\,\frac{e^{i(kr-\omega t)}}{r}$$

S2. Huygens-Fresnel Principle.

Christiaan Huygens (1629-1695)
Painting by Bernard Vaillant
Museum Hofwijck, Voorburg, The Netherlands

Huygens-Fresnel Principle says that you can replace a wave front with "baby waves" to get the wave of the future.

The baby wave:

$$\psi(r,t) = C\,\frac{e^{i(kr-\omega t)}}{r}$$

Augustin-Jean Fresnel (1788-1827).
From Frontispiece of his collected works (1866)
Image Courtesy Wikipedia

We will be analyzing diffraction patterns on a screen and we will be interested in the brightness averaged over time. So we can take the baby wave as follows.
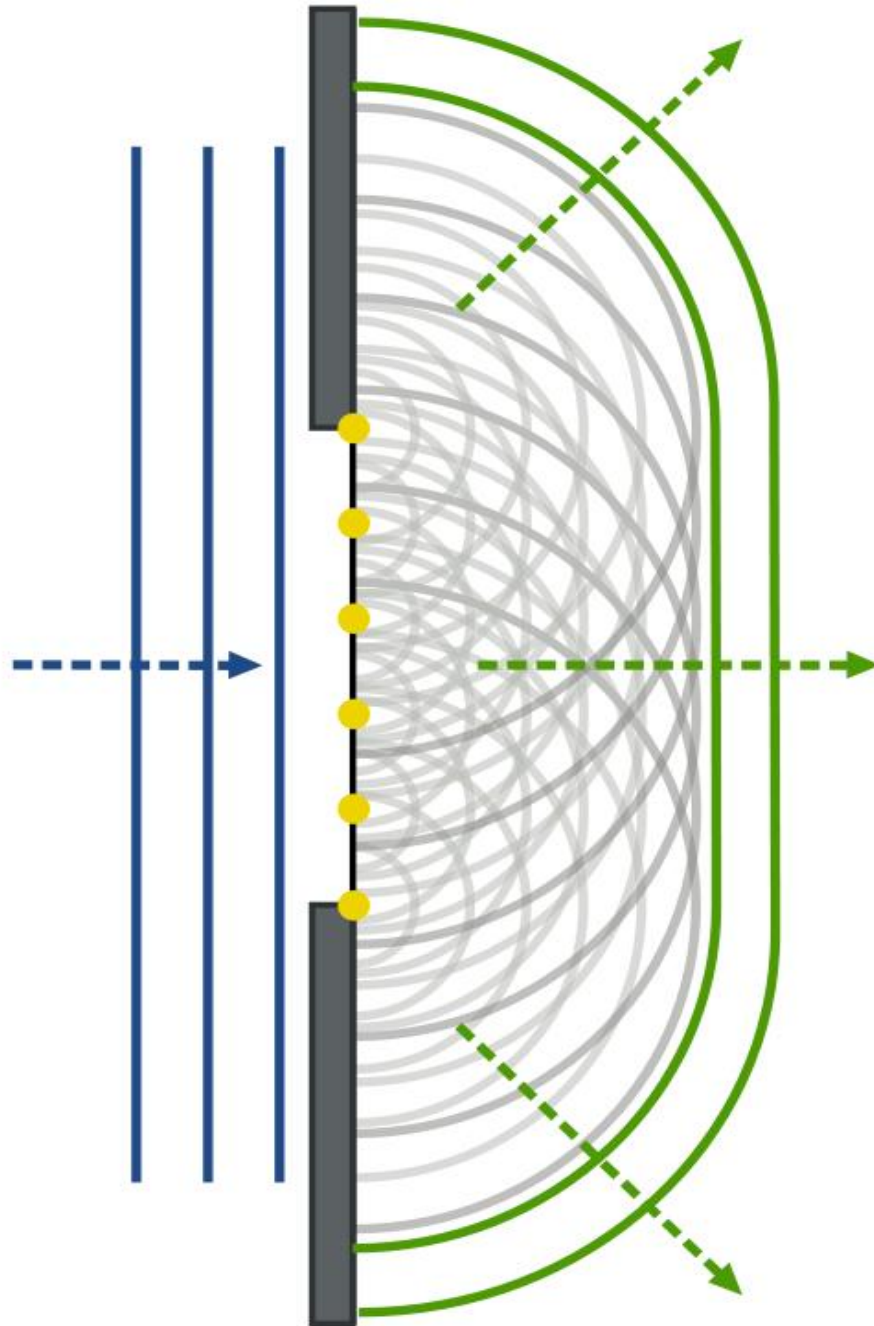
Baby Wave (time independent): $\quad C\,\dfrac{e^{ikr}}{r}$

Integrate over the source: $\quad \displaystyle\int C\,\frac{e^{ikr}}{r}\,dS$

Baby Waves – Integrate over the Opening Area

$$\int C \, \frac{e^{ikr}}{r} \, dA$$

The distance a baby-wave source to a screen far away is equal to r.
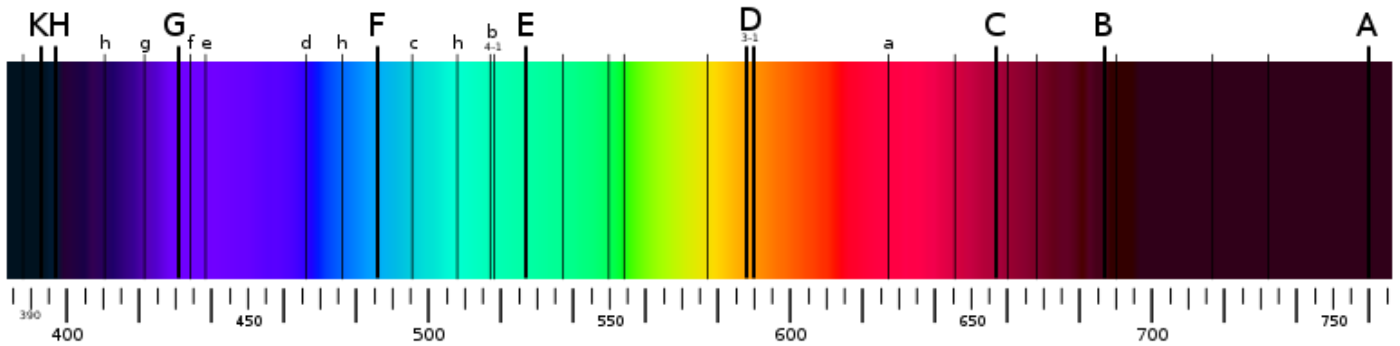
S3. Fraunhofer Diffraction.
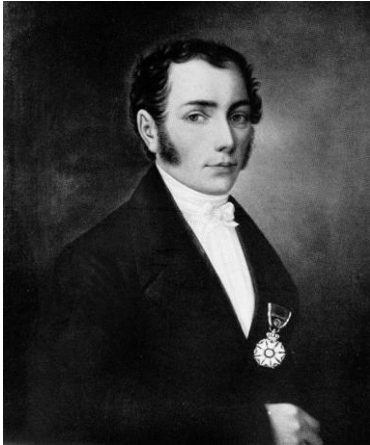
Joseph von Fraunhofer (1787-1826)
Physicist, Astronomer, Inventor, Lens Manufacturer

Invented the spectroscope.
Absorption spectra of stars called Fraunhofer lines in his honor.

The Solar Spectrum (Fraunhofer Lines)

Courtesy Wikipedia: Gebruiker:MaureenV. Released to the Public Domain.

Far-Field diffraction, i.e., the diffraction seen on a screen far away from a source, is named after Fraunhofer. Shine light at a small opening. The spatial part of the wave using the Huygens-Fresnel principle is
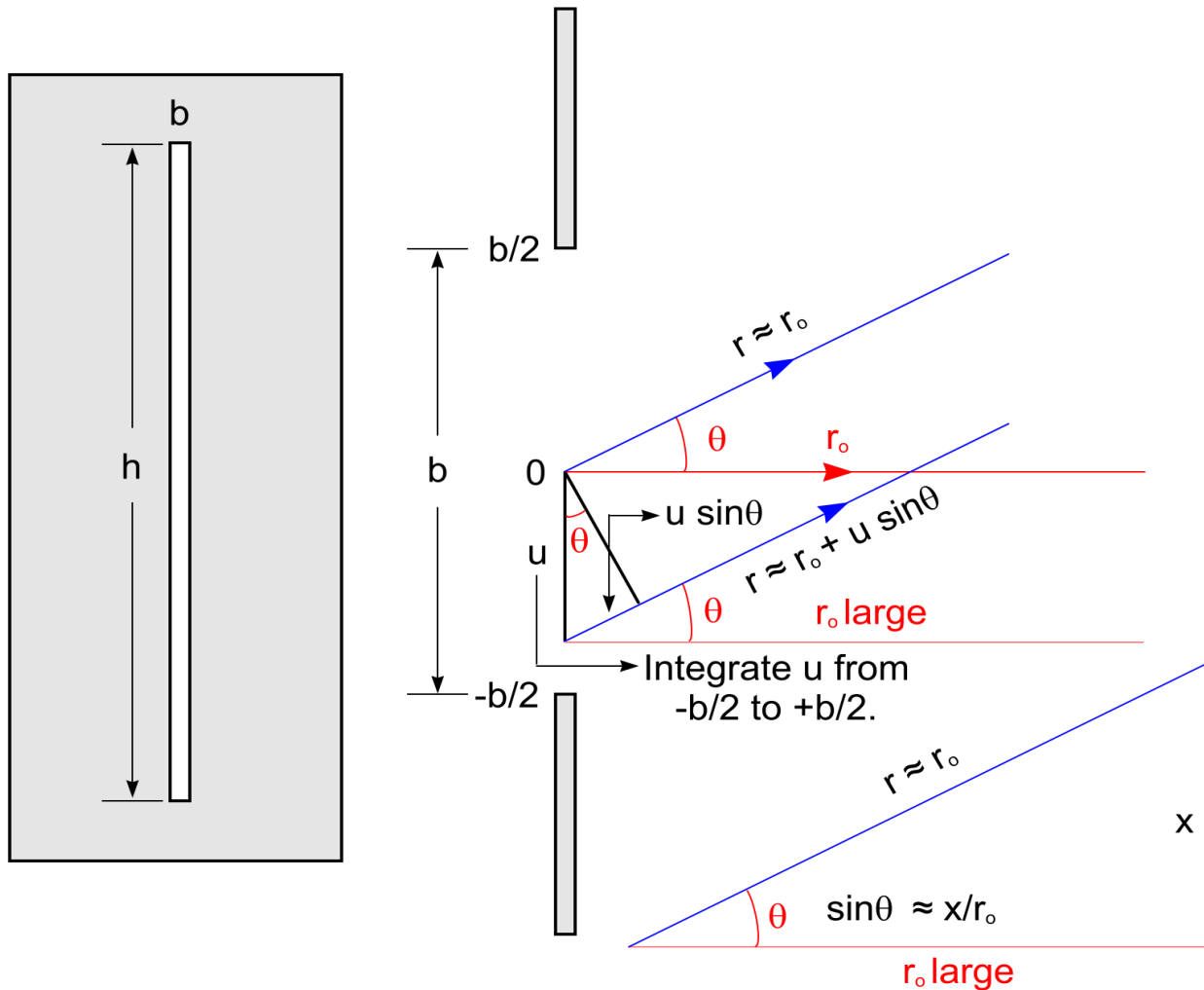
$$\psi(r) = E_o \frac{e^{ikr}}{r} \, .$$

But we need to integrate over the opening, i.e., aperture area to get the wave amplitude E_p at a point on the far away screen.

$$E_p = \int_A E_o \frac{e^{ikr}}{r} \, dA$$

Let the thin slit have dimension b for the width and the length (or height) be h.

S4. Single Slit.

The integral is then $E_p = \int\limits_A E_o \dfrac{e^{ikr}}{r} dA = \int\limits_{-b/2}^{b/2} E_o \dfrac{e^{ikr}}{r} h\,du$.

For large r we take $\dfrac{1}{r} = \dfrac{1}{r_o}$ but can't in e^{ikr} since we have trig functions that max out at 1.

Therefore $E_p = \dfrac{E_o h}{r_o} \int\limits_{-b/2}^{b/2} e^{ik(r_o + u\sin\theta)} du$. Redefine $C = \dfrac{E_o}{r_o}$.

$E_p = Ch \int\limits_{-b/2}^{b/2} e^{ik(r_o + u\sin\theta)} du$ => $E_p = Ch \int\limits_{-b/2}^{b/2} e^{ikr_o} e^{iku\sin\theta} du$

$$E_p = Che^{ikr_o} \int_{-b/2}^{b/2} e^{iku \sin \theta} du$$

$$E_p = Che^{ikr_o} \int_{-b/2}^{b/2} e^{(ik \sin \theta)u} du$$

$$E_p = Che^{ikr_o} \frac{e^{(ik \sin \theta)u}}{ik \sin \theta} \Big|_{-b/2}^{b/2}$$

$$E_p = \frac{Che^{ikr_o}}{ik \sin \theta} \left[e^{(ik \sin \theta)b/2} - e^{-(ik \sin \theta)b/2} \right]$$

$$E_p = \frac{Che^{ikr_o}}{ik \sin \theta} \left[e^{i\frac{1}{2}kb \sin \theta} - e^{-i\frac{1}{2}kb \sin \theta} \right]$$

It is customary to define $\beta = \dfrac{1}{2} kb \sin \theta$. Then

$$E_p = \frac{Che^{ikr_o}}{ik \sin \theta} \left[e^{i\beta} - e^{-i\beta} \right] \quad \Rightarrow \quad E_p = \frac{Che^{ikr_o}}{(k \sin \theta / 2)} \left[\frac{e^{i\beta} - e^{-i\beta}}{2i} \right]$$

$$E_p = \frac{Che^{ikr_o}}{(k \sin \theta / 2)} \sin \beta \quad \Rightarrow \quad E_p = \frac{Chbe^{ikr_o}}{(kb \sin \theta / 2)} \sin \beta$$

$$E_p = Chbe^{ikr_o} \frac{\sin \beta}{\beta} \qquad \text{The area of the slit is } hb \equiv A.$$

Irradiance $I = \dfrac{1}{2} E_p E_p^* = \dfrac{1}{2} \left[CAe^{ikr_o} \dfrac{\sin \beta}{\beta} \right] \left[CAe^{ikr_o} \dfrac{\sin \beta}{\beta} \right]^*$.

$$I = \frac{1}{2}\left[CAe^{ikr_o}\frac{\sin\beta}{\beta}\right]\left[CAe^{-ikr_o}\frac{\sin\beta}{\beta}\right] = \frac{1}{2}C^2A^2(\frac{\sin\beta}{\beta})^2$$

$$I = I_o(\frac{\sin\beta}{\beta})^2 = I_o\text{sinc}^2\beta$$
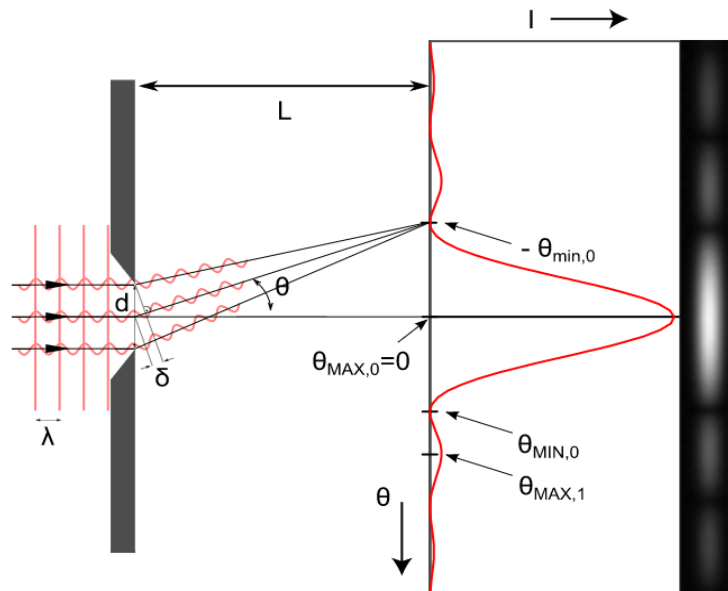
But the max occurs when $\beta = \frac{1}{2}kb\sin\theta = 0$ for the sinc function.

$$\beta = 0 \quad \Rightarrow \quad \theta = 0$$

Therefore we have the two following equivalent expressions, where $\beta = (kb\sin\theta)/2$.

$$\boxed{I(\theta) = I(0)\text{sinc}^2\beta} \qquad \boxed{I(\theta) = I(0)\frac{\sin^2\beta}{\beta^2}}$$

Single-Slit Diffraction

In the figure you see bright and dark regions, places of constructive and destructive interference respectively. Where do the extrema occur.

$$\boxed{I(\theta) = I(0)\frac{\sin^2\beta}{\beta^2}} \qquad \boxed{\beta = \frac{1}{2}kb\sin\theta}$$

(a) The minima. The angles for these are easy since we want $I(\theta)=0$. These places of destructive inference occur when the sine function hits 0, except for the 0° case since the sinc function is 1 there. But all the other usual places of zero for the sine function apply.
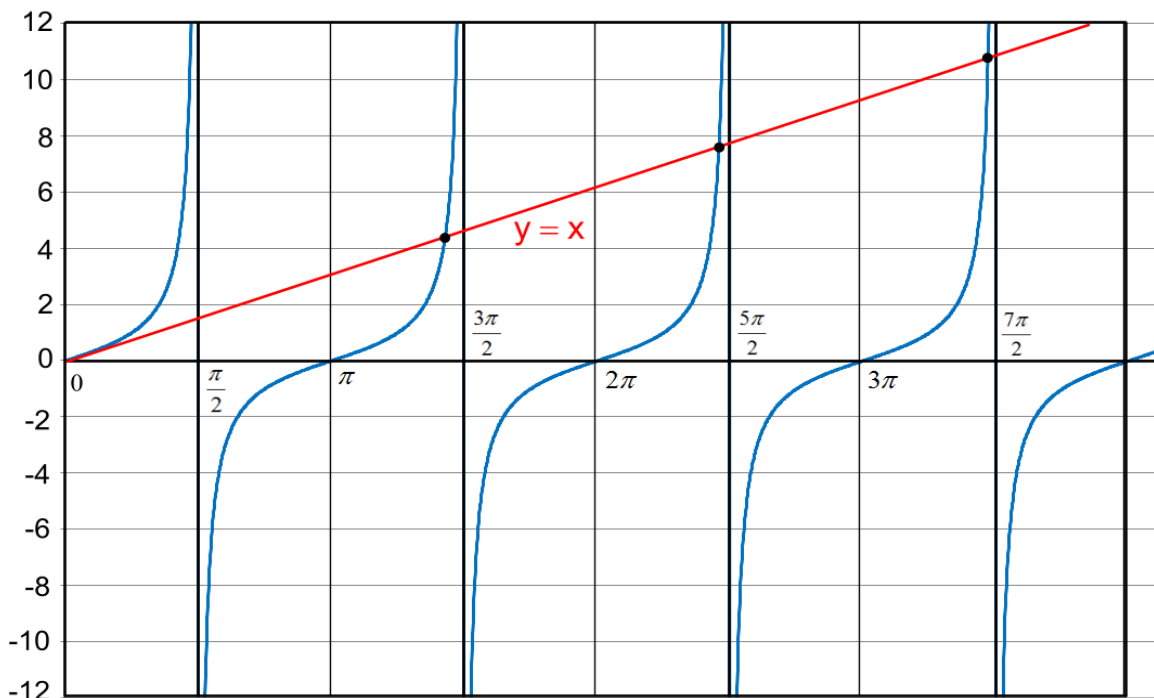
$$\beta = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$$

(b) The maxima. Now we do the max-min problem on $\dfrac{\sin \beta}{\beta}$.

$$\frac{d}{d\beta}(\frac{\sin \beta}{\beta}) = -\frac{1}{\beta^2}\sin \beta + \frac{1}{\beta}\cos \beta = 0 \quad \Rightarrow \quad \frac{1}{\beta^2}\sin \beta = \frac{1}{\beta}\cos \beta$$

$$\Rightarrow \quad \frac{\sin \beta}{\cos \beta} = \beta \quad \Rightarrow \quad \tan \beta = \beta \qquad \text{A transcendental equation!}$$

The function y = tan(x) superimposed with the function y = x

The above graph is a sketch. Here is how to find the values.
Set your calculator to radians and use the graph as a guide as follows.

Let the case where the graphs join at zero be $x_0 = 0$. If you hit tan for 0 you get 0.

Call the next case 1: $x_1 = 1.4\ \pi$ from the graph since $3\pi/2 = 1.5\ \pi$. Try $1.4\ \pi$, $1.42\ \pi$, etc. and you will find $x_1 = 1.43\ \pi = 4.49$ is very close. Similarly, try $x_2 = 2.4\ \pi$ since $5\pi/2 = 2.5\ \pi$. You can jump around a little and find that $x_2 = 2.46\ \pi = 7.73$ is very close. Trick exploration gives you an intuitive feel of the transcendental equation.

An efficient way to find the roots is to resort to online WolframAlpha and enter tan x = x. The first few solutions are
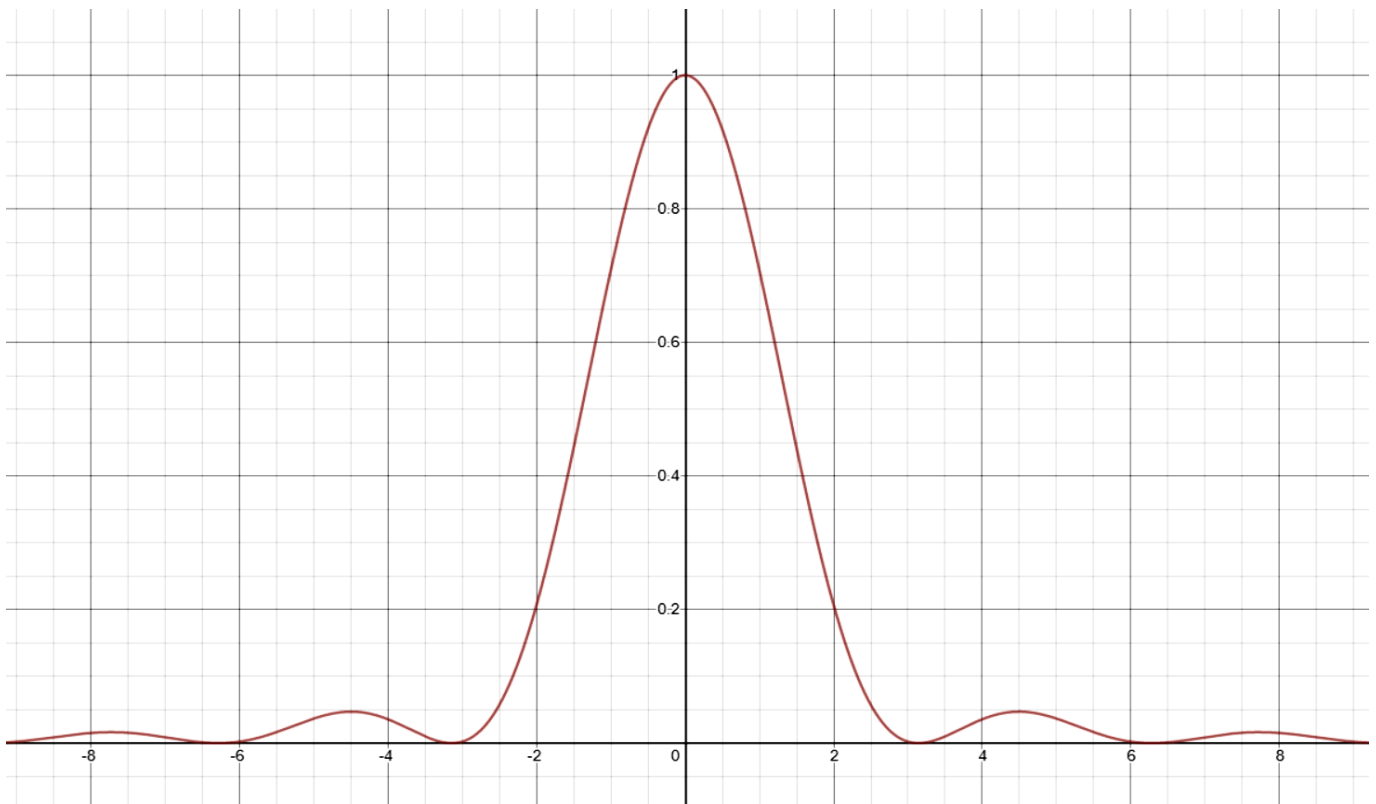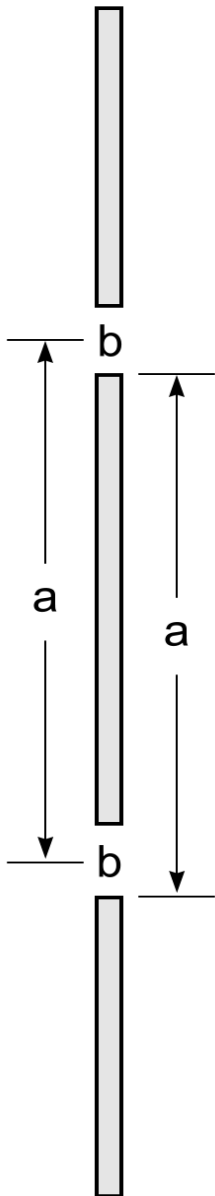
$$x_0 = 0$$

$$x_1 = \pm 4.493...$$

$$x_2 = \pm 7.725...$$

Can you find this below in the plot of $\text{sinc}^2(x)$?

Plot of $\text{sinc}^2(x)$ from desmos.com

S5. Double Slit.

Our previous integral $E_p = Ch \displaystyle\int_{-b/2}^{b/2} e^{ik(r_o + u\sin\theta)} du$ can be adapted easily.

Since the screen is so far away, we can slip the zero reference down to the bottom of the lower slit so that

$$\frac{E_p}{Ch} = \int_0^b e^{ik(r_o + u\sin\theta)} du + \int_a^{a+b} e^{ik(r_o + u\sin\theta)} du .$$

$$\boxed{\frac{E_p}{Ch} e^{-ikr_o} = \int_0^b e^{iku\sin\theta} du + \int_a^{a+b} e^{iku\sin\theta} du}$$

$$\frac{E_p}{Ch} e^{-ikr_o} = \frac{e^{iku\sin\theta}}{ik\sin\theta}\Big|_0^b + \frac{e^{iku\sin\theta}}{ik\sin\theta}\Big|_a^{a+b}$$

$$\frac{E_p}{Ch} e^{-ikr_o} ik\sin\theta = e^{ikb\sin\theta} - 1 + e^{ik(a+b)\sin\theta} - e^{ika\sin\theta}$$

$$\frac{E_p}{Ch} e^{-ikr_o} ik\sin\theta = e^{ikb\sin\theta} - 1 + e^{ika\sin\theta} e^{ikb\sin\theta} - e^{ika\sin\theta}$$

$$\frac{E_p}{Ch} e^{-ikr_o} ik\sin\theta = e^{ikb\sin\theta} - 1 + e^{ika\sin\theta}(e^{ikb\sin\theta} - 1)$$

$$\frac{E_p}{Ch} e^{-ikr_o} ik\sin\theta = (e^{ikb\sin\theta} - 1)(1 + e^{ika\sin\theta})$$

$$\frac{E_p}{Ch} e^{-ikr_o} ik\sin\theta = e^{ikb\sin\theta/2}(e^{ikb\sin\theta/2} - e^{-ikb\sin\theta/2})(1 + e^{ika\sin\theta})$$

$$\frac{E_p}{Ch}e^{-ikr_o}ik\sin\theta = e^{ikb\sin\theta/2}\left[2i\sin\frac{kb\sin\theta}{2}\right](1+e^{ika\sin\theta})$$

$$\frac{E_p}{Ch}e^{-ikr_o}ik\sin\theta = e^{ikb\sin\theta/2}\left[2i\sin\frac{kb\sin\theta}{2}\right]e^{ika\sin\theta/2}(e^{-ika\sin\theta/2}+e^{ika\sin\theta/2})$$

$$\frac{E_p}{Ch}e^{-ikr_o}ik\sin\theta = e^{ikb\sin\theta/2}\left[2i\sin\frac{kb\sin\theta}{2}\right]e^{ika\sin\theta/2}\left[2\cos\frac{ka\sin\theta}{2}\right]$$

Recall our $\beta = (kb\sin\theta)/2$. Now add $\alpha = (ka\sin\theta)/2$.

$$\frac{E_p}{Ch}e^{-ikr_o}ik\sin\theta = e^{i\beta}\left[2i\sin\beta\right]e^{i\alpha}\left[2\cos\alpha\right]$$

$$E_p = \frac{Ch}{ik\sin\theta}e^{ikr_o}e^{i\alpha}e^{i\beta}4i\cos\alpha\sin\beta$$

$$E_p = \frac{4Ch}{k\sin\theta}e^{ikr_o}e^{i\alpha}e^{i\beta}\cos\alpha\sin\beta$$

$$E_p = \frac{2Chb}{kb\sin\theta/2}e^{ikr_o}e^{i\alpha}e^{i\beta}\cos\alpha\sin\beta$$

Recall the area of the slit is $hb \equiv A$.

$$E_p = (2CA)e^{ikr_o}e^{i\alpha}e^{i\beta}\cos\alpha\frac{\sin\beta}{\beta}$$

Irradiance $I = \frac{1}{2}E_p E_p^*$

$$= \frac{1}{2} \left[(2CA)e^{ikr_o} e^{i\alpha} e^{i\beta} \cos\alpha \frac{\sin\beta}{\beta} \right] \left[(2CA)e^{ikr_o} e^{i\alpha} e^{i\beta} \cos\alpha \frac{\sin\beta}{\beta} \right]^*$$

$$= \frac{1}{2} (2CA)^2 \left[e^{ikr_o} e^{i\alpha} e^{i\beta} \cos\alpha \frac{\sin\beta}{\beta} \right] \left[e^{-ikr_o} e^{-i\alpha} e^{-i\beta} \cos\alpha \frac{\sin\beta}{\beta} \right]$$

$$= \frac{1}{2} (2CA)^2 e^{ikr_o} e^{-ikr_o} e^{i\alpha} e^{-i\alpha} e^{i\beta} e^{-i\beta} \cos^2\alpha \frac{\sin^2\beta}{\beta^2}$$

$$= \frac{1}{2} (2CA)^2 \cdot 1 \cdot 1 \cdot 1 \cdot \cos^2\alpha \frac{\sin^2\beta}{\beta^2}$$

$$I(\theta) = I_{max} \cos^2\alpha \frac{\sin^2\beta}{\beta^2}$$

The max occurs when $\theta = 0°$.

Then, $\beta = \dfrac{kb\sin\theta}{2} \to 0$

and

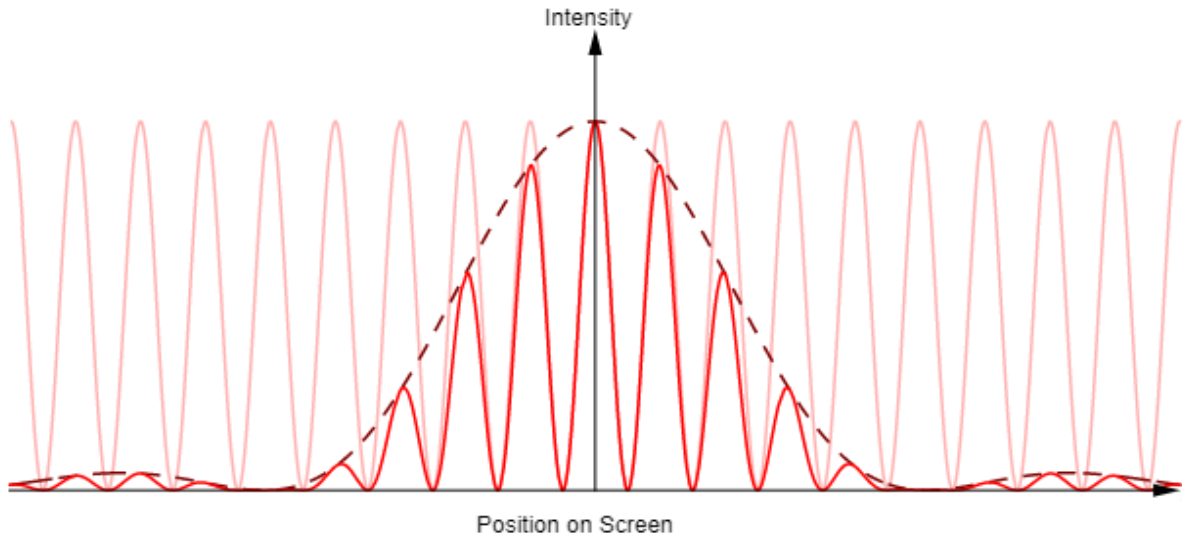$\alpha = \dfrac{ka\sin\theta}{2} \to 0$.

With $\dfrac{\sin\beta}{\beta} \to 1$ and $\cos\alpha \to 1$. Finally, $I(0°) = I_{max} \cdot 1 \cdot 1$.
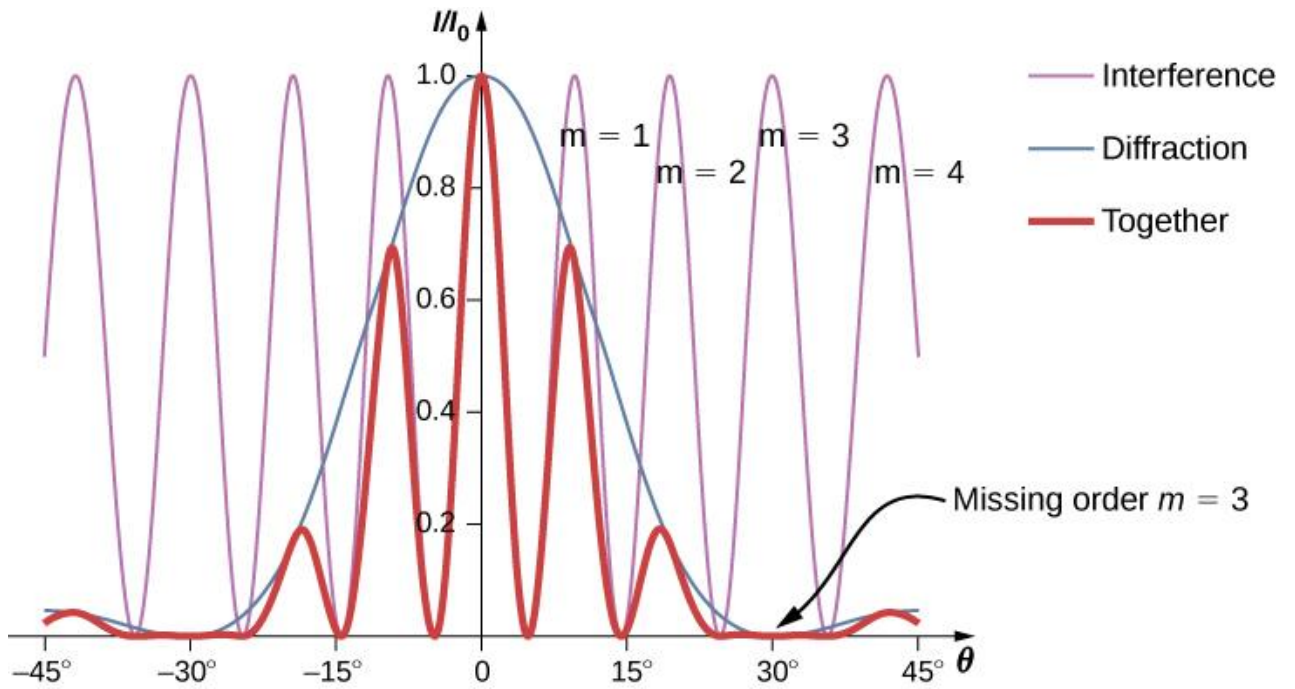
$$\boxed{I(\theta) = I(0)\cos^2\alpha \frac{\sin^2\beta}{\beta^2}}$$

Interference Due to Two Slits $\cos^2\alpha$ (faint red), Diffraction $\text{sinc}^2\beta$ (dashed), and solid red $I(\theta)$.

Courtesy GeoGebra. Author Dave Nero. Creative Commons

Same as Above with Missing Orders Labeled

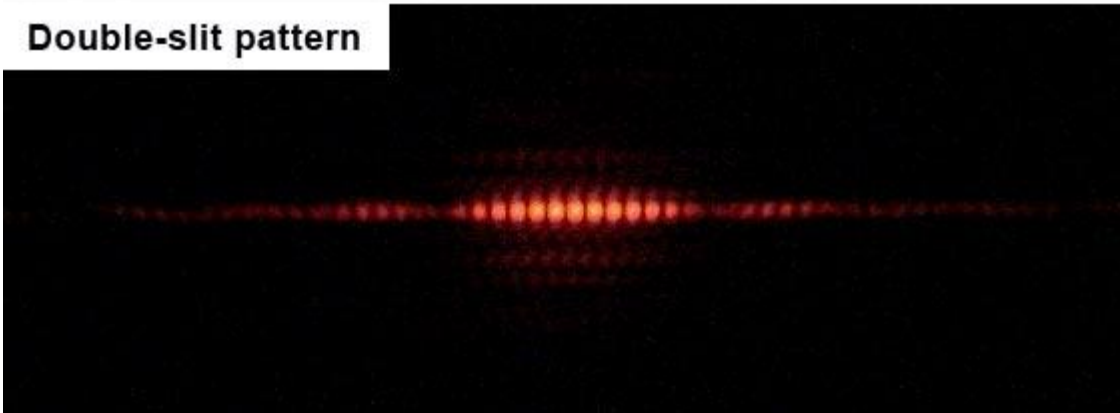Courtesy LibreTexts, UC Davis. OpenStax University Physics. Creative Commons

Single Slit:

$$I(\theta) = I(0)\frac{\sin^2 \beta}{\beta^2} \qquad \beta = \frac{1}{2}kb\sin\theta$$

Single-slit pattern

Double-slit pattern

Wikipedia: Jordgette. Creative Commons

Double Slit:

$$I(\theta) = I(0)\cos^2 \alpha \, \frac{\sin^2 \beta}{\beta^2}$$

$$\alpha = \frac{ka\sin\theta}{2} \qquad \beta = \frac{1}{2}kb\sin\theta$$