HW S1. Geometric Series. Show that

$$S_n = 1 + r + r^2 + r^3 + ... + r^n = \frac{1 - r^{n+1}}{1 - r}$$

Then apply your general formula to show the following result you will need in Problem HW S2.

$$1 + e^{i\alpha} + e^{2i\alpha} = \frac{e^{3i\alpha} - 1}{e^{i\alpha} - 1}$$

Solution.

$$S_n = 1 + r + r^2 + r^3 + ... + r^n$$

$$rS_n = r + r^2 + r^3 + r^4 ... + r^{n+1}$$

$$S_n - rS_n = 1 - r^{n+1}$$

$$S_n(1 - r) = 1 - r^{n+1}$$

$$S_n = \frac{1 - r^{n+1}}{1 - r}$$

Let $r = e^{i\theta}$. Then $S_2 = 1 + r + r^2 = 1 + e^{i\gamma} + e^{2i\gamma}$.

Using our S_n formula with $n = 2$, $S_2 = 1 + r + r^2 = \frac{1 - r^3}{1 - r} = \frac{r^3 - 1}{r - 1}$, giving

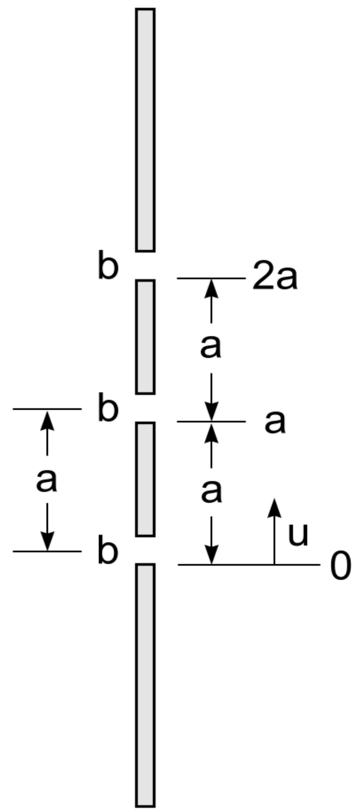
$$\boxed{1 + e^{i\gamma} + e^{2i\gamma} = \frac{e^{3i\gamma} - 1}{e^{i\gamma} - 1}}.$$

HW S2. Triple-Slit Diffraction. Adapt the single-slit Fraunhofer diffraction formula

$$E_p = Ch \int_{-b/2}^{b/2} e^{ik(r_o + u\sin\theta)} du$$

to a the triple slit shown in the figure. Define

$$\alpha = \frac{ka\sin\theta}{2} \quad \text{and} \quad \beta = \frac{kb\sin\theta}{2}.$$



Show that the irradiance is given by the formula below.

$$I = I_o \left[\frac{\sin(\beta)}{\beta}\right]^2 \left[\frac{\sin(3\alpha)}{3\sin\alpha}\right]^2$$

Justify that indeed $I_o = I(0°)$ using L'Hôpital's Rule.

Solution.

$$\frac{E_p}{Ch} e^{-ikr_o} = \int_0^b e^{iku\sin\theta} du + \int_a^{a+b} e^{iku\sin\theta} du + \int_{2a}^{2a+b} e^{iku\sin\theta} du$$

$$\frac{E_p}{Ch} e^{-ikr_o} = \left.\frac{e^{iku\sin\theta}}{ik\sin\theta}\right|_0^b + \left.\frac{e^{iku\sin\theta}}{ik\sin\theta}\right|_a^{a+b} + \left.\frac{e^{iku\sin\theta}}{ik\sin\theta}\right|_{2a}^{2a+b}$$

$$\frac{E_p}{Ch} e^{-ikr_o} ik\sin\theta =$$

$$(e^{ikb\sin\theta} - 1) + (e^{ik(a+b)\sin\theta} - e^{ika\sin\theta}) + (e^{ik(2a+b)\sin\theta} - e^{ik2a\sin\theta})$$

$$= e^{ikb\sin\theta} - 1 + e^{ika\sin\theta} e^{ikb\sin\theta} - e^{ika\sin\theta} + e^{ik2a\sin\theta} e^{ikb\sin\theta} - e^{i2ka\sin\theta}$$

$$= e^{2i\beta} - 1 + e^{2i\alpha} e^{2i\beta} - e^{2i\alpha} + e^{4i\alpha} e^{2i\beta} - e^{4i\alpha}$$

$$\frac{E_p}{Ch} e^{-ikr_o} ik \sin\theta = e^{2i\beta} - 1 + e^{2i\alpha}(e^{2i\beta} - 1) + e^{4i\alpha}(e^{2i\beta} - 1)$$

$$\frac{E_p}{Ch} e^{-ikr_o} ik \sin\theta = (e^{2i\beta} - 1)(1 + e^{2i\alpha} + e^{4i\alpha})$$

$$\frac{E_p}{Ch} e^{-ikr_o} ik \sin\theta = e^{i\beta}(e^{i\beta} - e^{-i\beta})(1 + e^{2i\alpha} + e^{4i\alpha})$$

$$\frac{E_p}{Ch} e^{-ikr_o} ik \sin\theta = e^{i\beta}(2i \sin\beta)(1 + e^{2i\alpha} + e^{4i\alpha})$$

Now use from HW S1: $1 + e^{i\gamma} + e^{2i\gamma} = \dfrac{e^{3i\gamma} - 1}{e^{i\gamma} - 1}$.

Let $\gamma = 2\alpha$. Then, $1 + e^{2i\alpha} + e^{4i\alpha} = \dfrac{e^{6i\gamma} - 1}{e^{2i\gamma} - 1}$

$$\frac{E_p}{Ch} e^{-ikr_o} ik \sin\theta = e^{i\beta}(2i \sin\beta)\left[\frac{e^{6i\gamma} - 1}{e^{2i\gamma} - 1}\right]$$

Note in general: $e^{i\theta} - 1 = e^{i\theta/2}(e^{i\theta/2} - e^{-i\theta/2})$

$$e^{i\theta} - 1 = e^{i\theta/2}(2i \sin\frac{\theta}{2})$$

Using this result twice in the last factor with the brackets gives us

$$\frac{E_p}{Ch} e^{-ikr_o} ik \sin\theta = e^{i\beta}(2i \sin\beta)\left[\frac{e^{3i\alpha}\, 2i \sin(3\alpha)}{e^{i\alpha}\, 2i \sin(\alpha)}\right]$$

$$\frac{E_p}{Ch} e^{-ikr_o} ik \sin\theta = e^{i\beta} e^{2i\alpha}(2i \sin\beta)\left[\frac{\sin(3\alpha)}{\sin(\alpha)}\right]$$

$$\frac{E_p}{Ch} e^{-ikr_o} \frac{k \sin\theta}{2} = e^{i\beta} e^{2i\alpha} \sin\beta \left[\frac{\sin(3\alpha)}{\sin\alpha} \right]$$

$$\frac{E_p}{Chb} e^{-ikr_o} \frac{kb \sin\theta}{2} = e^{i\beta} e^{2i\alpha} \sin\beta \left[\frac{\sin(3\alpha)}{\sin\alpha} \right]$$

$$\frac{E_p}{CA} e^{-ikr_o} \beta = e^{i\beta} e^{2i\alpha} \sin\beta \left[\frac{\sin(3\alpha)}{\sin\alpha} \right]$$

$$E_p = CAe^{ikr_o} e^{i\beta} e^{2i\alpha} \frac{\sin\beta}{\beta} \left[\frac{\sin(3\alpha)}{\sin\alpha} \right]$$

$$I = \frac{1}{2} E_p E_p^* = I = \frac{1}{2} (CA)^2 e^{ikr_o} e^{i\beta} e^{2i\alpha} e^{-ikr_o} e^{-i\beta} e^{-2i\alpha} \frac{\sin^2\beta}{\beta^2} \left[\frac{\sin^2(3\alpha)}{\sin^2\alpha} \right]$$

$$I = \frac{1}{2} (CA)^2 \frac{\sin^2\beta}{\beta^2} \left[\frac{\sin^2(3\alpha)}{\sin^2\alpha} \right]$$

Normalization to find the result when $\theta = 0°$. When $\theta = 0°$,

$$\alpha = \frac{ka \sin 0°}{2} = 0 \quad \text{and} \quad \beta = \frac{kb \sin 0°}{2} = 0 .$$

Now use L'Hôpital's Rule.

$$\lim_{\beta \to 0} \frac{\sin\beta}{\beta} = \lim_{\beta \to 0} \frac{\cos\beta}{1} = 1 \quad \text{and} \quad \lim_{\alpha \to 0} \frac{\sin(3\alpha)}{\alpha} = \lim_{\beta \to 0} \frac{3\cos(3\alpha)}{1} = 3 .$$

$$\boxed{I = I_o \frac{\sin^2\beta}{\beta^2} \left[\frac{\sin(3\alpha)}{3\sin\alpha} \right]^2}$$