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T1. Diffraction Grating.

Single Slit

$$E_p = Che^{ikr_o} \int_{-b/2}^{b/2} e^{iku\sin\theta} du \qquad C = \frac{E_o}{r_o}$$

$$I(\theta) = I(0)\frac{\sin^2\beta}{\beta^2} \qquad \beta = \frac{1}{2}kb\sin\theta$$

Double Slit

$$\frac{E_p}{Ch}e^{-ikr_o} = \int_0^b e^{iku\sin\theta} du + \int_a^{a+b} e^{iku\sin\theta} du$$

$$I(\theta) = I(0)\cos^2\alpha\,\frac{\sin^2\beta}{\beta^2}$$

$$\alpha = (ka\sin\theta)/2 \qquad \beta = (kb\sin\theta)/2$$

N Slits

Refer to the figure at the left. Note that the last integral for 7 slits starts at 6a, i.e., (N − 1) a.

$$\frac{E_p}{Ch}e^{-ikr_o} = \int_0^b e^{iku\sin\theta} du + \int_a^{a+b} e^{iku\sin\theta} du + \int_{2a}^{2a+b} e^{iku\sin\theta} du + ... + \int_{(N-1)a}^{(N-1)a+b} e^{iku\sin\theta} du$$

$$\frac{E_p}{Ch}e^{-ikr_o} = \frac{e^{iku\sin\theta}}{ik\sin\theta}\Big|_0^b + \frac{e^{iku\sin\theta}}{ik\sin\theta}\Big|_a^{a+b} + \frac{e^{iku\sin\theta}}{ik\sin\theta}\Big|_{2a}^{2a+b} + ... + \frac{e^{iku\sin\theta}}{ik\sin\theta}\Big|_{(N-1)a}^{(N-1)a+b}$$

$$\frac{E_p}{Ch}e^{-ikr_o} = \frac{e^{ikb\sin\theta}-1}{ik\sin\theta} + \frac{e^{ik(a+b)\sin\theta}-e^{ika\sin\theta}}{ik\sin\theta}$$

$$+\frac{e^{ik(2a+b)\sin\theta}-e^{ik2a\sin\theta}}{ik\sin\theta} + ... + \frac{e^{ik[(N-1)a+b]\sin\theta}-e^{ik(N-1)a\sin\theta}}{ik\sin\theta}$$

$$ik\sin\theta\frac{E_p}{Ch}e^{-ikr_o} = e^{ikb\sin\theta}-1+e^{ik(a+b)\sin\theta}-e^{ika\sin\theta}$$

$$+e^{ik(2a+b)\sin\theta}-e^{ik2a\sin\theta} + ... + e^{ik[(N-1)a+b]\sin\theta}-e^{ik(N-1)a\sin\theta}$$

$$ik\sin\theta\frac{E_p}{Ch}e^{-ikr_o} = e^{ikb\sin\theta}-1+e^{ika\sin\theta}[e^{ikb\sin\theta}-1]$$

$$+e^{ik2a\sin\theta}[e^{ikb\sin\theta}-1] + ... + e^{ik[(N-1)a}[e^{ikb\sin\theta}-1]$$

$$ik\sin\theta\frac{E_p}{Ch}e^{-ikr_o} = (e^{ikb\sin\theta}-1)[1+e^{ika\sin\theta}+e^{ik2a\sin\theta}+...+e^{ik(N-1)a\sin\theta}]$$

Geometric Series with Math Notation: $S_n \equiv 1 + r + r^2 + r^3 + ... + r^n$

$$rS_n = r + r^2 + r^3 + r^4 + ... + r^{n+1} \quad => \quad S_n - rS_n = 1 - r^{n+1} \quad =>$$

$$\boxed{S_n = \frac{1-r^{n+1}}{1-r}} \quad \text{or} \quad \boxed{S_n = \frac{r^{n+1}-1}{r-1}}$$

Let $r = e^{ika\sin\theta}$ and $n = N-1$

$$ik \sin\theta \frac{E_p}{Ch} e^{-ikr_o} = (e^{ikb\sin\theta} - 1)(\frac{e^{ikNa\sin\theta} - 1}{e^{ika\sin\theta} - 1})$$

Note: $e^{i\phi} - 1 = e^{i\phi/2}\left(e^{i\phi/2} - e^{-i\phi/2}\right) = e^{i\phi/2}[2i\sin(\phi/2)]$.

We have three appearances of this kind of arrangement in our equation.

$$ik \sin\theta \frac{E_p}{Ch} e^{-ikr_o} = e^{ikb(\sin\theta)/2}(2i)\sin(\frac{kb\sin\theta}{2}) \frac{e^{ikNa(\sin\theta)/2}(2i\sin\frac{kNa\sin\theta}{2})}{e^{ika(\sin\theta)/2}(2i\sin\frac{ka\sin\theta}{2})}$$

Let $\alpha = (ka\sin\theta)/2$ and $\beta = (kb\sin\theta)/2$ as we did before.

$$ik \sin\theta \frac{E_p}{Ch} e^{-ikr_o} = e^{i\beta}(2i)\sin(\beta) \frac{e^{iN\alpha}\sin(N\alpha)}{e^{i\alpha}\sin\alpha}$$

$$E_p = \frac{2Ch}{k\sin\theta} e^{ikr_o} e^{i\beta}\sin(\beta) \frac{e^{iN\alpha}\sin(N\alpha)}{e^{i\alpha}\sin\alpha}$$

$$\beta = (kb\sin\theta)/2 \quad => \quad \frac{k\sin\theta}{2} = \frac{\beta}{b} \quad => \quad \frac{2}{k\sin\theta} = \frac{b}{\beta}$$

$$E_p = Chbe^{ikr_o} e^{i\beta} \frac{\sin(\beta)}{\beta} \frac{e^{iN\alpha}\sin(N\alpha)}{e^{i\alpha}\sin\alpha}$$

$$E_p = CAe^{-ikr_o} e^{i\beta} e^{i(N-1)\alpha} \frac{\sin(\beta)}{\beta} \frac{\sin(N\alpha)}{\sin\alpha} \quad \text{where } A = bh$$

$$I = \frac{E_p E_p^*}{2} = \frac{C^2 A^2}{2} \frac{\sin^2(\beta)}{\beta^2} \frac{\sin^2(N\alpha)}{\sin^2\alpha}$$

What happens at $\theta = 0°$? First note that for $\theta \to 0°$ we have $\alpha \to 0°$ and $\beta \to 0°$.

Remember that $\lim_{\beta \to 0} \dfrac{\sin(\beta)}{\beta} = 1$ by L'Hospital's (L'Hôpital's) rule.

By L'Hospital's rule $\lim_{\alpha \to 0} \dfrac{\sin(N\alpha)}{\alpha} = \lim_{\alpha \to 0} \dfrac{N\cos(N\alpha)}{1} = N$

Therefore $I = I(0°) \dfrac{\sin^2(\beta)}{\beta^2} \dfrac{\sin^2(N\alpha)}{N^2 \sin^2 \alpha}$

$$\boxed{I = I_o \left[\dfrac{\sin(\beta)}{\beta} \right]^2 \left[\dfrac{\sin(N\alpha)}{N \sin \alpha} \right]^2}$$

Note that for the double slit: $\dfrac{\sin(N\alpha)}{N \sin \alpha} \to \dfrac{\sin(2\alpha)}{2\sin \alpha} = \dfrac{2 \sin \alpha \cos \alpha}{2 \sin \alpha} = \cos \alpha$,

and then $I = I_o \left[\dfrac{\sin(\beta)}{\beta} \right]^2 \left[\cos \alpha \right]^2$ as we found last class.

When we did the two-slit problem we found that the envelope $\dfrac{\sin^2(\beta)}{\beta^2}$ of the diffraction

pattern modulated the maxima coming from the interference pattern $\cos^2 \alpha$. The interference pattern produces the principal maxima. These principal maxima occur at $\pm m\pi$, where $m = 0, \pm 1, \pm 2, ...$ because

$$\lim_{\alpha \to m\pi} \dfrac{\sin(N\alpha)}{N\alpha} = \lim_{\alpha \to m\pi} \dfrac{N\cos(N\alpha)}{N} = \pm 1$$

Secondary maxima occur when $N\alpha = \pm \dfrac{\pi}{2}, \pm \dfrac{3\pi}{2}, \pm \dfrac{5\pi}{2}, \cdots$ because the sine is 1 there.
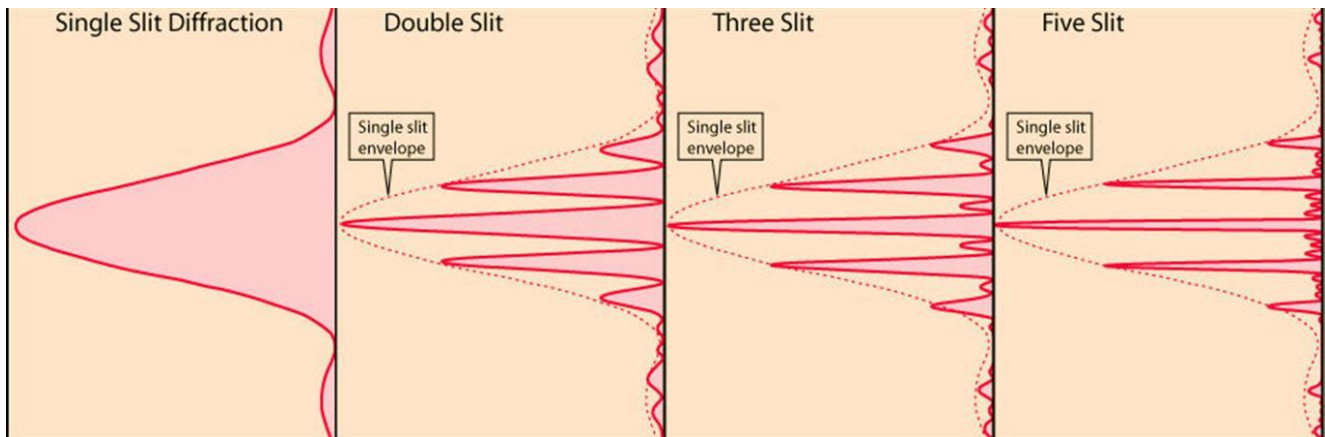
Let $N\alpha = \pm p \dfrac{\pi}{2}$, where p is odd. For gratings N will be large. Therefore,

$$\frac{\sin(N\alpha)}{N\sin\alpha} = \frac{\sin(\dfrac{p\pi}{2})}{N\sin(\dfrac{p\pi}{2N})} \approx \pm \frac{1}{N(\dfrac{p\pi}{2N})} = \pm \frac{2}{p\pi}$$

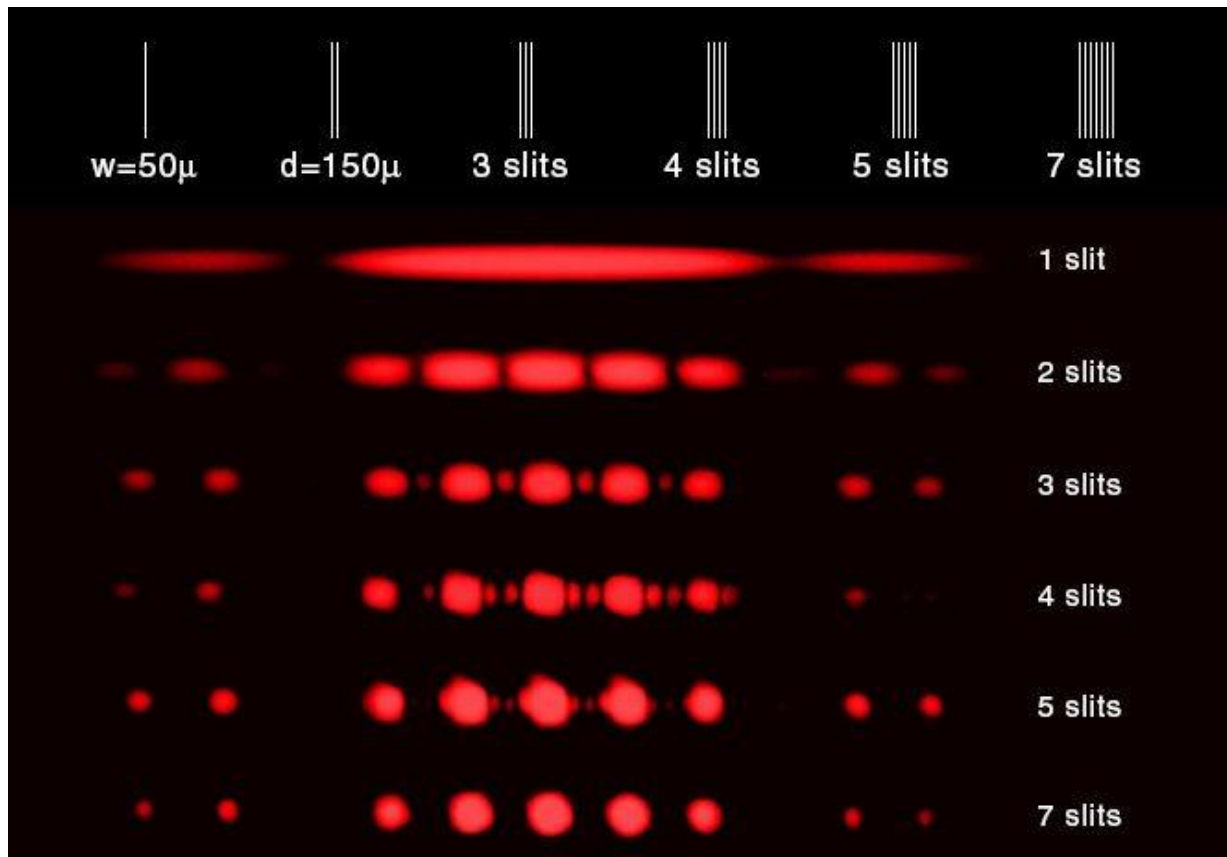$$\left[\frac{\sin(N\alpha)}{N\sin\alpha}\right]^2 = \left[\frac{2}{p\pi}\right]^2 < 1$$

These maxima are called secondary because they get dimmer and dimmer as p increases.

Diffraction Patterns for N = 1, N = 2, N = 3, and N = 5.

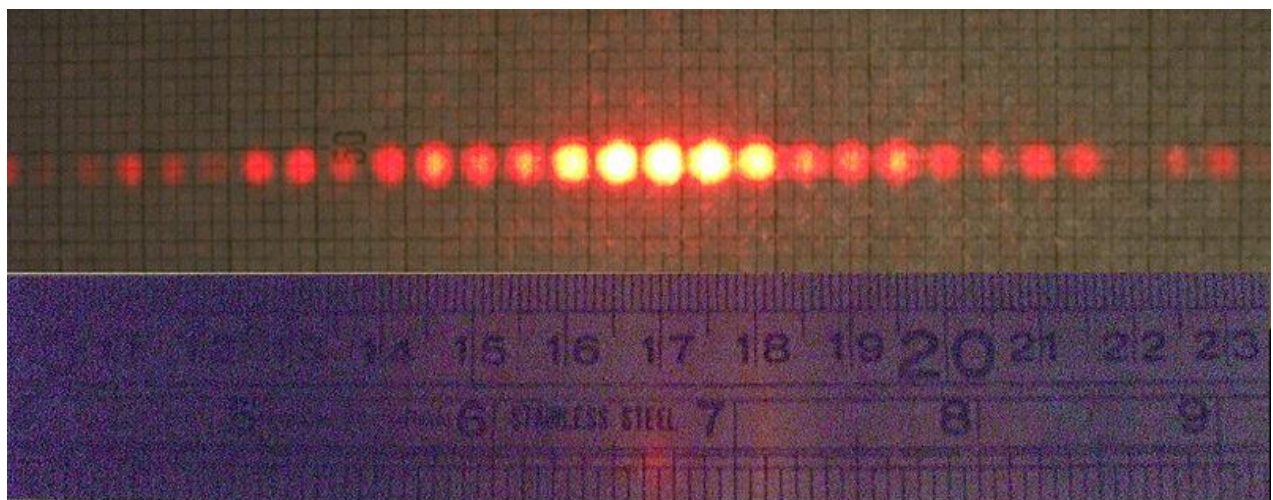

Courtesy physics.stackexchange.com. Creative Commons

Diffraction Patterns

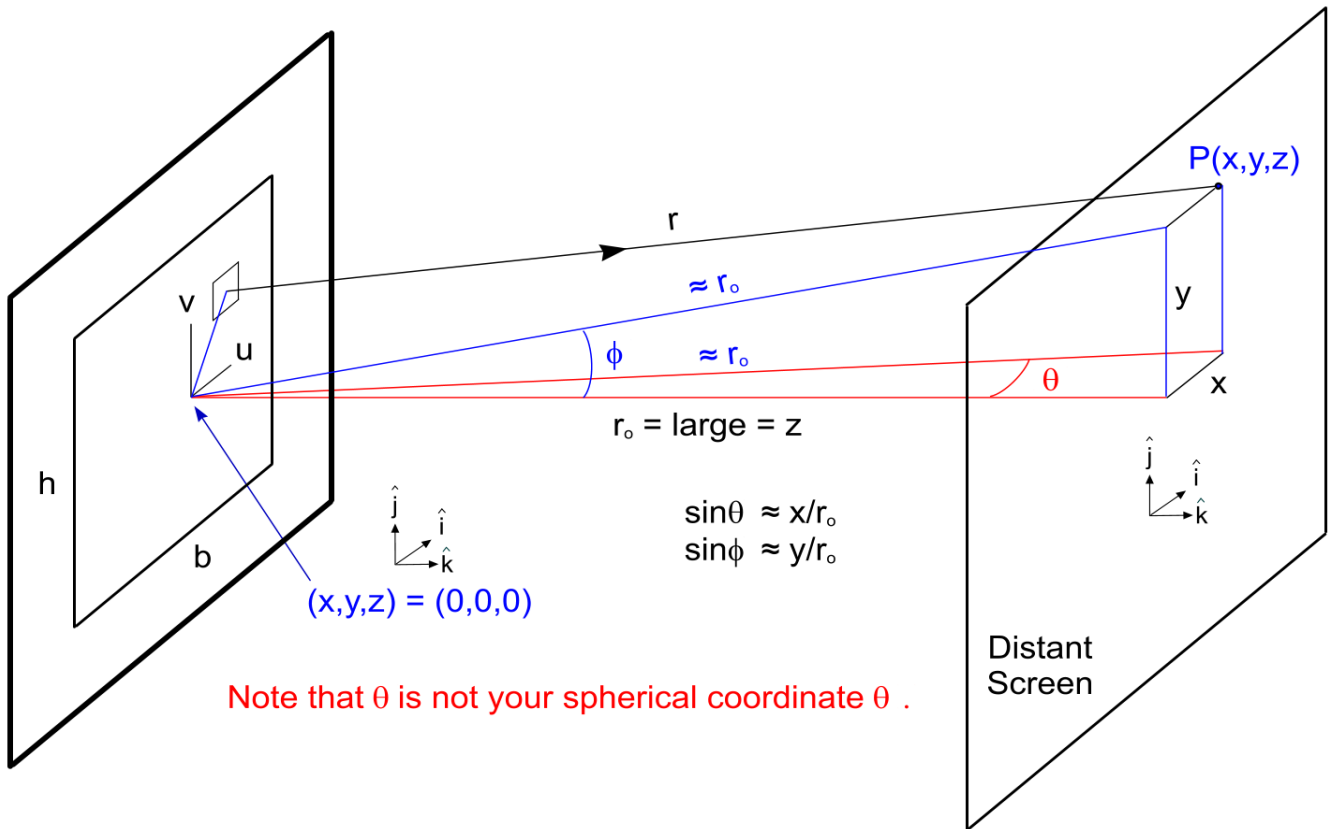Courtesy physicsopenlab.org. Creative Commons

Note the nice uniform pattern in the central region for the above 7-slit case. The diffraction grating has many slits. Check out the result below for N = 150 slits, b = 0.0625 mm, and a = 0.25 mm with 633 nm light from a helium-neon (He-Ne) laser. Ben Gurion Physics Department.

Wikipedia: Authors Shim'on and Slava Rybka. Creative Commons

T2. Rectangular Aperture.

Apply $E_p = \int\limits_{\text{Aperture}} E_o \frac{e^{ikr}}{r} dA = \frac{E_o}{r_o} \int\limits_A e^{ikr} dA = C \int\limits_A e^{ikr} dA$ to the rectangular aperture.

r_o = large = z

$\sin\theta \approx x/r_o$
$\sin\phi \approx y/r_o$

$(x,y,z) = (0,0,0)$

Note that θ is not your spherical coordinate θ .

Distant Screen

$$r = \sqrt{(x-u)^2 + (y-v)^2 + (z-0)^2}$$

$$r = \sqrt{(x^2 - 2xu + u^2) + (y^2 - 2yv + v^2) + z^2}$$

$$r = \sqrt{x^2 + y^2 + z^2 - 2xu - 2yv + u^2 + v^2}$$

Note that $x << z$, $y << z$, $u << z$, and $v << z$.

$$r = \sqrt{z^2 - 2xu - 2yv}$$

$$r_o = z \quad \Rightarrow \quad r = \sqrt{r_o^2 - 2xu - 2yv} = r_o\sqrt{1 - \frac{2xu}{r_o^2} - \frac{2yv}{r_o^2}}$$

Since $u \ll x$ and $v \ll y$ we have

$$r = r_o\left(1 - \frac{2xu}{r_o^2} - \frac{2yv}{r_o^2}\right)^{1/2} \approx r_o\left(1 - \frac{xu}{r_o^2} - \frac{yv}{r_o^2}\right).$$

$$r = r_o - \frac{xu}{r_o} - \frac{yv}{r_o}$$

$$r = r_o - u\sin\theta - v\sin\phi$$

$$E_p = C\int_A e^{ikr}\,dA = C\int_A e^{ik(r_o - u\sin\theta - v\sin\phi)}\,dA$$

$$E_p = Ce^{ikr_o}\int_A e^{-ik(u\sin\theta + v\sin\phi)}\,dA$$

$$E_p = Ce^{ikr_o}\int_{-b/2}^{b/2} e^{-iku\sin\theta}\,du \int_{-h/2}^{h/2} e^{-ikv\sin\phi}\,dv$$

Let $u \to -u$ and $v \to -v$

$$E_p = Ce^{ikr_o}\int_{+b/2}^{-b/2} e^{iku\sin\theta}\,du \int_{+h/2}^{-h/2} e^{ikv\sin\phi}\,dv = Ce^{ikr_o}\int_{-b/2}^{b/2} e^{iku\sin\theta}\,du \int_{-h/2}^{h/2} e^{ikv\sin\phi}\,dv$$

Each integral is exactly in the form of what we did before.

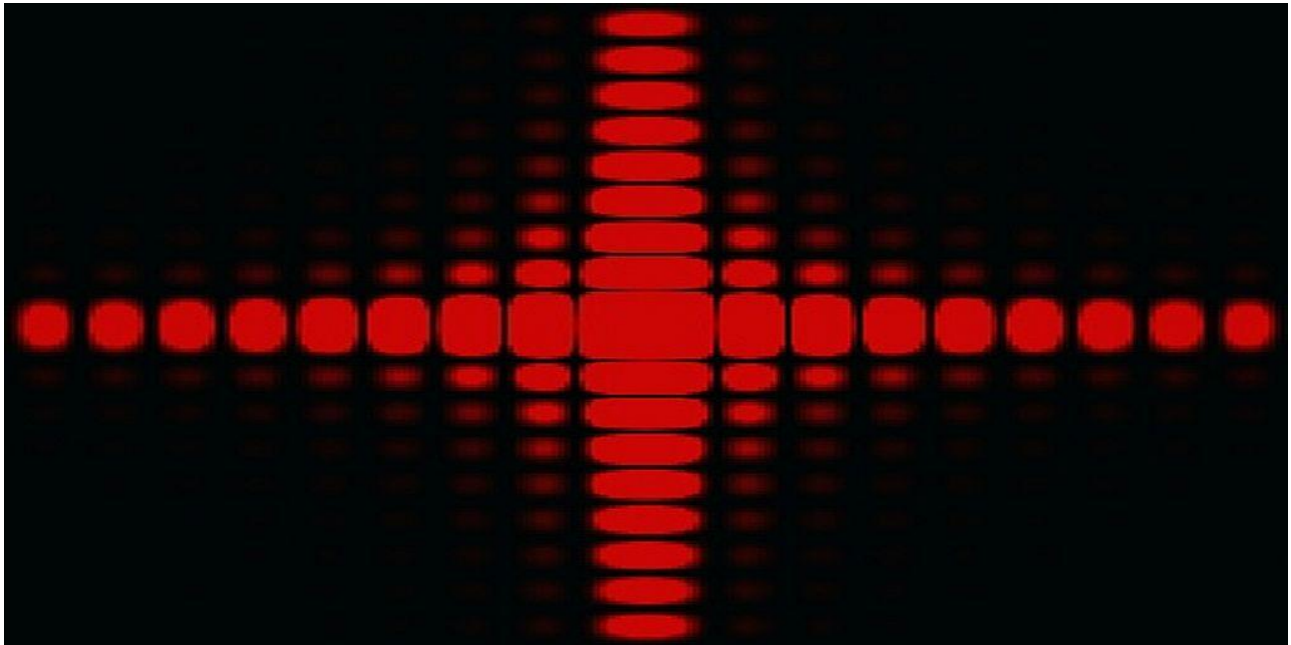$$E_p = Che^{ikr_o}\int_{-b/2}^{b/2} e^{iku\sin\theta}\,du$$

We did the one-dimensional version earlier and defined $\beta = \dfrac{1}{2} kd \sin\theta$.

Now add $\alpha = \dfrac{1}{2} kh \sin\phi$ and we get

$$I(\theta) = I(0)\left[\frac{\sin\alpha}{\alpha}\right]^2 \left[\frac{\sin\beta}{\beta}\right]^2 \qquad \boxed{\alpha = \frac{1}{2} kh \sin\phi} \qquad \boxed{\beta = \frac{1}{2} kd \sin\theta}$$
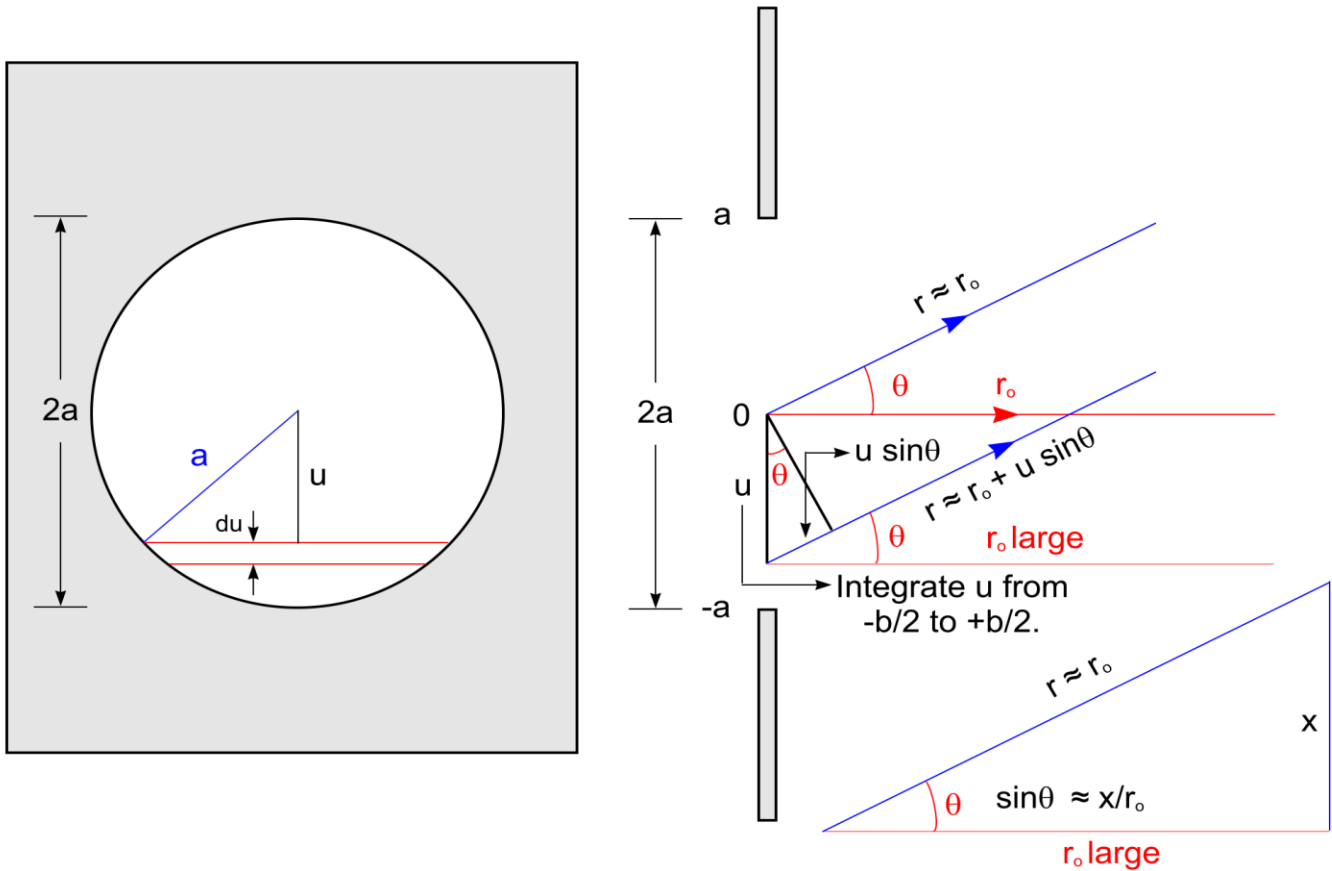

Wikipedia: Epzcaw. Creative Commons

$$\text{Note } E_p = Che^{ikr_o} \int_{-b/2}^{b/2} e^{-iku\sin\theta}\,du \quad \text{where } \sin\theta = \frac{x}{r_o}$$

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-b/2}^{b/2} e^{-ikx}\,du \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)e^{-ikx}\,du$$

A Fourier Transform! The Fourier transform is visualized at the screen!
Remember the Fourier transform of the rectangular pulse gave the sinc in transform space?
We found the electric field amplitude at the screen for a uniformly-lit single slit to be the sinc!

T3. Circular Aperture.

Apply $E_p = \int_A E_o \dfrac{e^{ikr}}{r} dA = \dfrac{E_o}{r_o} \int_A e^{ikr} dA = C \int_A e^{ikr} dA$ to the circular aperture.

We use $\quad r = r_o + u\sin\theta \quad$ and $\quad dA = 2\sqrt{a^2 - u^2}\, du$.

$$E_p = C\int_A e^{ikr} dA = C \int_{-a}^{a} e^{ik(r_o + u\sin\theta)} 2\sqrt{a^2 - u^2}\, du$$

$$E_p = 2Ce^{ikr_o} \int_{-a}^{a} e^{iku\sin\theta} \sqrt{a^2 - u^2}\, du = 2Ce^{ikr_o} \int_{-a}^{a} e^{iku\sin\theta} a\sqrt{1 - (u/a)^2}\, du$$

Let $s = \dfrac{u}{a}$. Then $ds = \dfrac{du}{a}$ and $-1 \le s \le 1$ as $-a \le u \le a$.

$$E_p = 2Cae^{ikr_o} \int_{-1}^{1} e^{ikas\sin\theta} \sqrt{1-s^2}\, a\, ds$$

$$E_p = 2Ca^2 e^{ikr_o} \int_{-1}^{1} e^{ikas\sin\theta} \sqrt{1-s^2}\, ds$$

Now recall our fondness for definitions like $\alpha = \dfrac{1}{2} kh\sin\phi$.

So let $\sigma = \dfrac{1}{2} k(2a)\sin\theta = ka\sin\theta$ since 2a is the aperture diameter.

We also choose a Greek letter for the angle, σ (sigma) to match our s variable.

Then $E_p = \dfrac{2}{\pi} CAe^{ikr_o} \int_{-1}^{1} e^{i\sigma s} \sqrt{1-s^2}\, ds$, where $A = \pi r^2$.

Define a new constant C' so that

$$E_p = C' \int_{-1}^{1} e^{i\sigma s} \sqrt{1-s^2}\, ds .$$

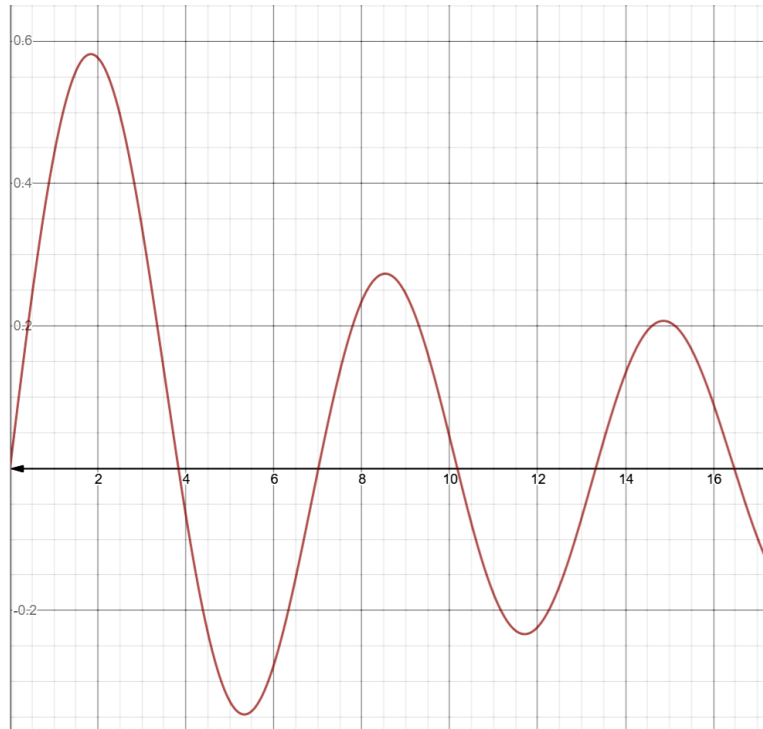Now we come to a dead end since we cannot simplify further.

We encounter here one of the Bessel functions of the 1st kind $J_1(\sigma)$:

$$\int_{-1}^{1} e^{i\sigma s} \sqrt{1-s^2}\, ds = \dfrac{\pi J_1(\sigma)}{\sigma}, \text{ giving } E_p = C' \dfrac{\pi J_1(\sigma)}{\sigma}.$$
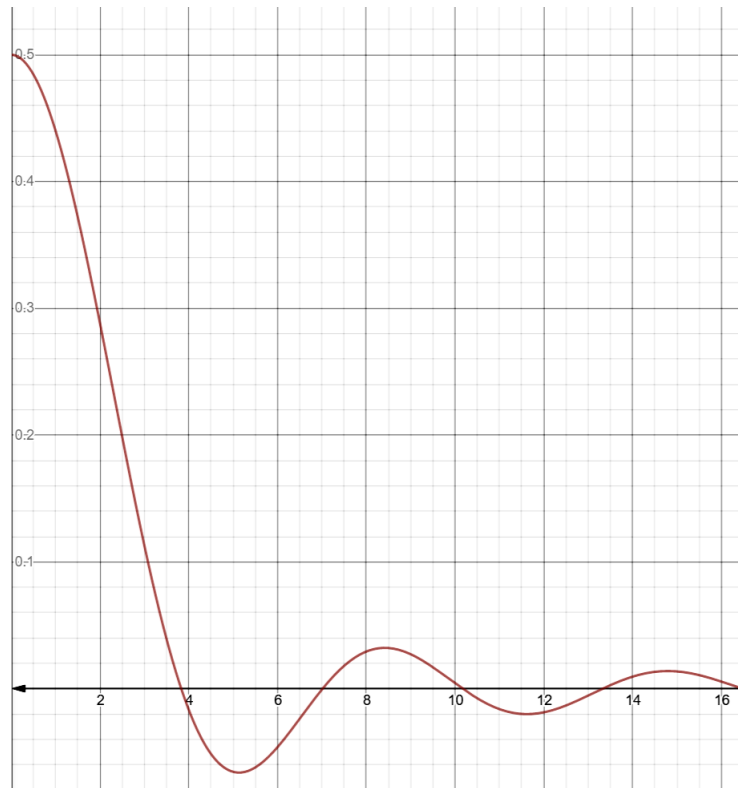
$$\int_{-1}^{1} e^{i\sigma s} \sqrt{1-s^2}\, ds = \int_{-1}^{1} [\cos(\sigma s) + i\sin(\sigma s)]\sqrt{1-s^2}\, ds = \int_{-1}^{1} \cos(\sigma s)\sqrt{1-s^2}\, ds$$

since the product of an odd and even function over a symmetric interval is zero.

$$J_1(x)$$

$$J_1(x) \, / \, x$$

Note that $\lim_{x \to 0}[J_1(x) / x] = 1/2$

The we take $E_p = C' \dfrac{\pi J_1(\sigma)}{\sigma} \to E_o \left[\dfrac{2J_1(\sigma)}{\sigma} \right]$ so $E_p(0°) = E_o$.

Remember $\sigma = ka \sin \theta$ goes to zero when θ approaches zero.

Therefore, to normalize the irradiance, we must have $I = I_o \left[\dfrac{2J_1(\sigma)}{\sigma} \right]^2$.

Note that the irradiance has its first zero when σ is near 4.

More precisely, the first zero occurs at $\sigma = 3.832$.

Then $\sigma = ka \sin \theta$ leads to $\sigma = ka \sin \theta = 3.832$.

But $k = \dfrac{2\pi}{\lambda}$. Therefore, $\dfrac{2\pi}{\lambda} a \sin \theta = 3.832$ and

$$\frac{2\pi}{\lambda} a \sin \theta = 3.832$$

Since $d = 2a$, $\dfrac{\pi}{\lambda} d \sin \theta = 3.832$.

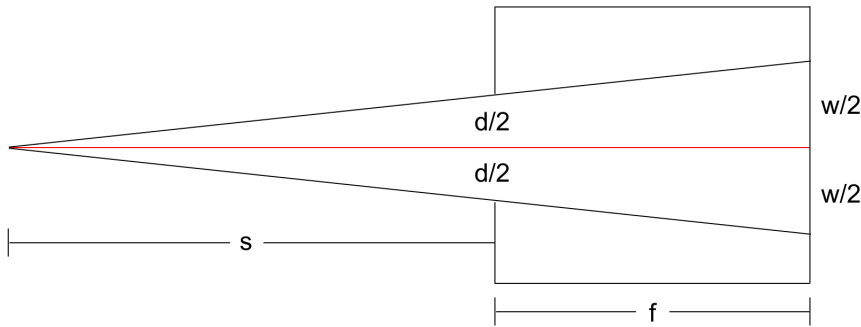$$d \sin \theta = 3.832 \frac{\lambda}{\pi}$$

The central bright region is called the Airy disk. It has a diameter

$$d \sin \theta = 1.22\lambda .$$

The sine of the angle that gets us to the edge of the Airy disk is $\sin \theta = 1.22 \dfrac{\lambda}{d}$

This result is also known as the **Rayleigh criterion**.

Remember pinhole camera resolution analysis for a diameter d?

We take $\sin\theta = \dfrac{w/2}{f}$,

since θ is measured from the center of the aperture to the edge of w/2. We take this sine to be our

$$\sin\theta = 1.22\frac{\lambda}{d} .$$

Note also that $d \approx w$.

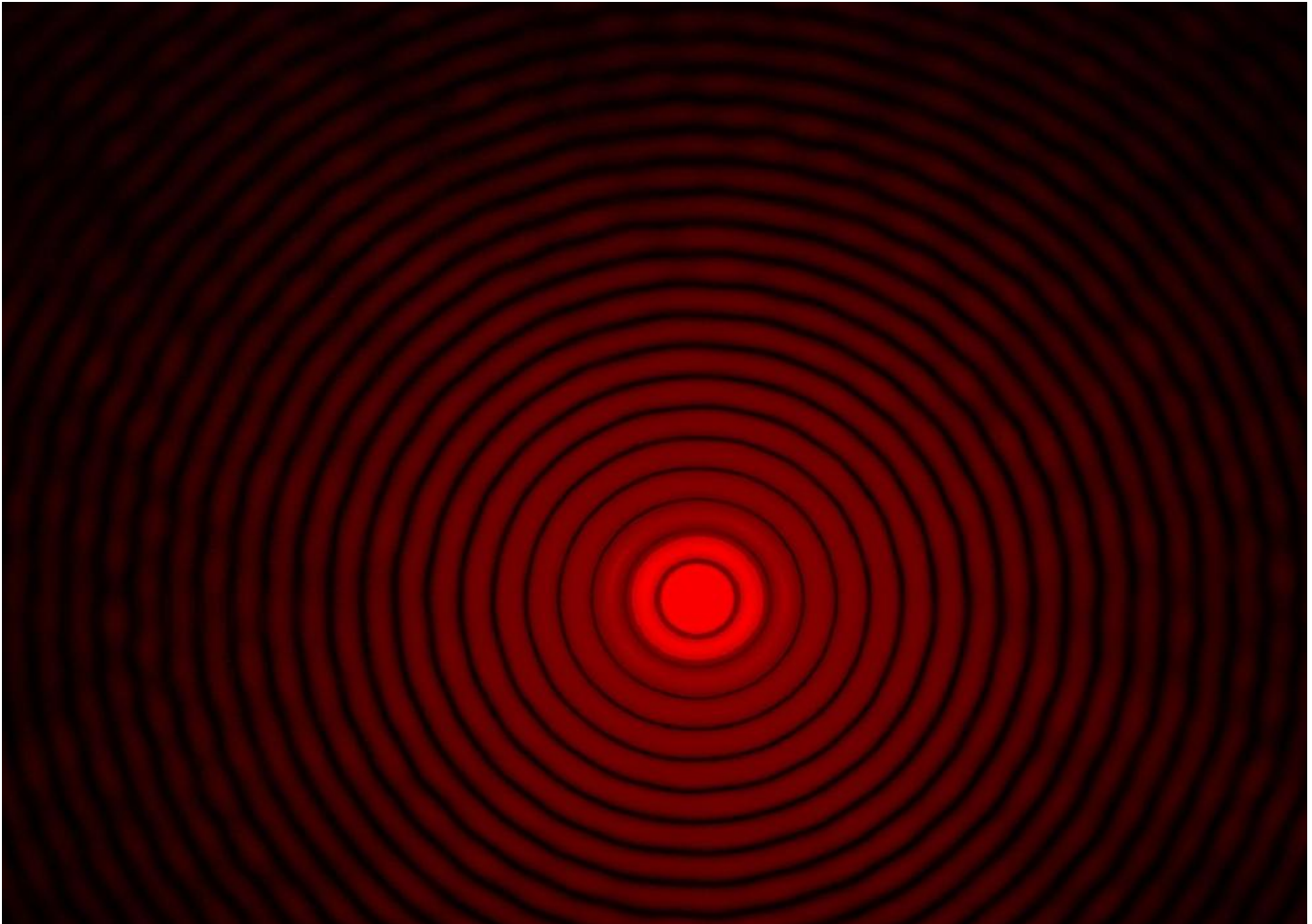Then, $\sin\theta = \dfrac{w/2}{f} = \dfrac{d/2}{f} = 1.22\dfrac{\lambda}{d} .$

$$\frac{d}{2} = 1.22\frac{\lambda}{d} f \quad\Rightarrow\quad d^2 = 2.44\lambda f$$

$$d = \sqrt{2.44\lambda f}$$

This formula was given in Chapter G. Now we see where it comes from.

Daniel Bernoulli 1700 – 1782	Friedrich Wilhelm Bessel 1784 - 1846	George Biddell Airy 1891 - 1892
Mathematician and Physicist	Astronomer, Mathematician Physicist	Mathematician and Astronomer
Defined what is known as the Bessel function.	Generalized the Bessel Function.	Airy Disk named after him.

Diffraction through Circular Pinhole Aperture with Airy Disk in the Center.

Wikipedia: Bautsch. Released into the Public Domain.

Pinhole size was 90 µm and 27 orders of diffraction are observed.