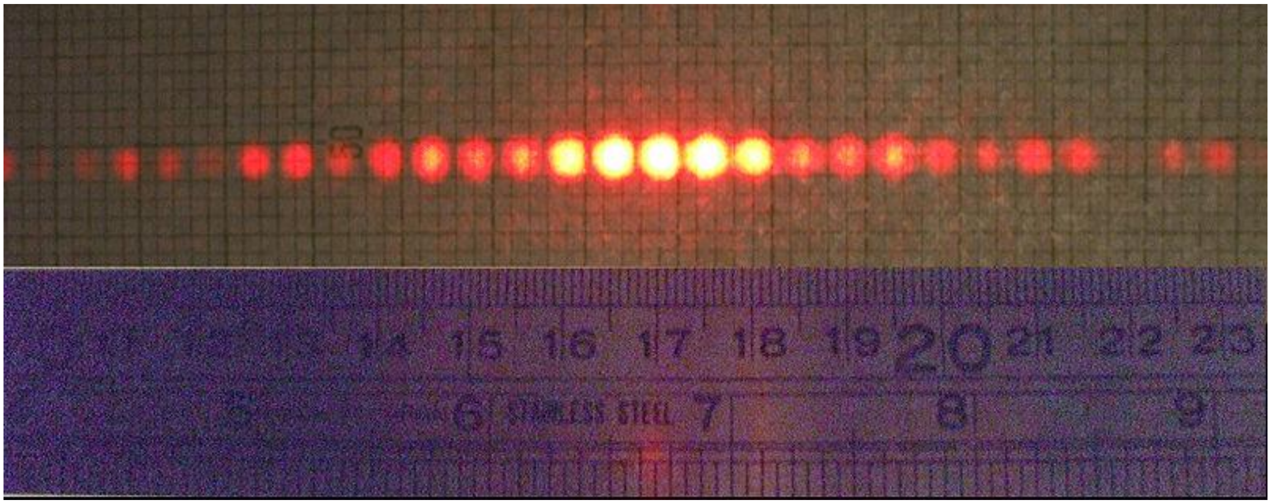
Modern Optics, Prof. Ruiz, UNCA doctorphys.com

Chapter T. Fraunhofer II – Homework SOLUTIONS

HW T1. Diffraction Grating. Below is a photo taken by the Physics Department at Ben Gurion University using a diffraction grating. The diffraction grating had N = 150 slits, b = 0.0625 mm (slit width), and a = 0.25 mm (distance between adjacent slit centers), with 632.8-nm light from a helium-neon (He-Ne) laser.



Wikipedia: Authors Shim'on and Slava Rybka. Creative Commons

Take the dark gap at 22 cm along the ruler to be the first missing order. Estimate the distance between the diffraction grating and the screen in the lab. Give your answer in meters to two significant figures.

For the first max to the right of the central maximum: $\sin\theta = \frac{x}{r_o} = \frac{(22-17)\text{ cm}}{r_o}$.

$$\sin\theta = \frac{5\text{ cm}}{r_o}$$

$$I = I_o\left[\frac{\sin(\beta)}{\beta}\right]^2\left[\frac{\sin(N\alpha)}{N\sin\alpha}\right]^2 \qquad \alpha = (ka\sin\theta)/2 \qquad \beta = (kb\sin\theta)/2$$

The closely spaced bright spots are due to the interference pattern, i.e., the factor $\left[\frac{\sin(N\alpha)}{N\sin\alpha}\right]^2$,

while the brightness is shaped by the diffraction pattern, i.e., the factor $\left[\frac{\sin(\beta)}{\beta}\right]^2$.

The first minimum in the diffraction factor occurs at $\beta = \pi$. Therefore,

$$\beta = (kb\sin\theta)/2 = \pi \quad \Rightarrow \quad kb\sin\theta = 2\pi \quad \Rightarrow \quad \frac{2\pi}{\lambda}b\sin\theta = 2\pi.$$

$$\frac{2\pi}{\lambda} b \sin\theta = 2\pi \quad => \quad \sin\theta = \frac{\lambda}{b}$$

Bringing back $\sin\theta = \frac{5 \text{ cm}}{r_o}$ gives $\frac{\lambda}{b} = \frac{5 \text{ cm}}{r_o}$

Now use $b = 0.0625 \text{ mm}$ and $\lambda = 632.8 \text{ nm}$.

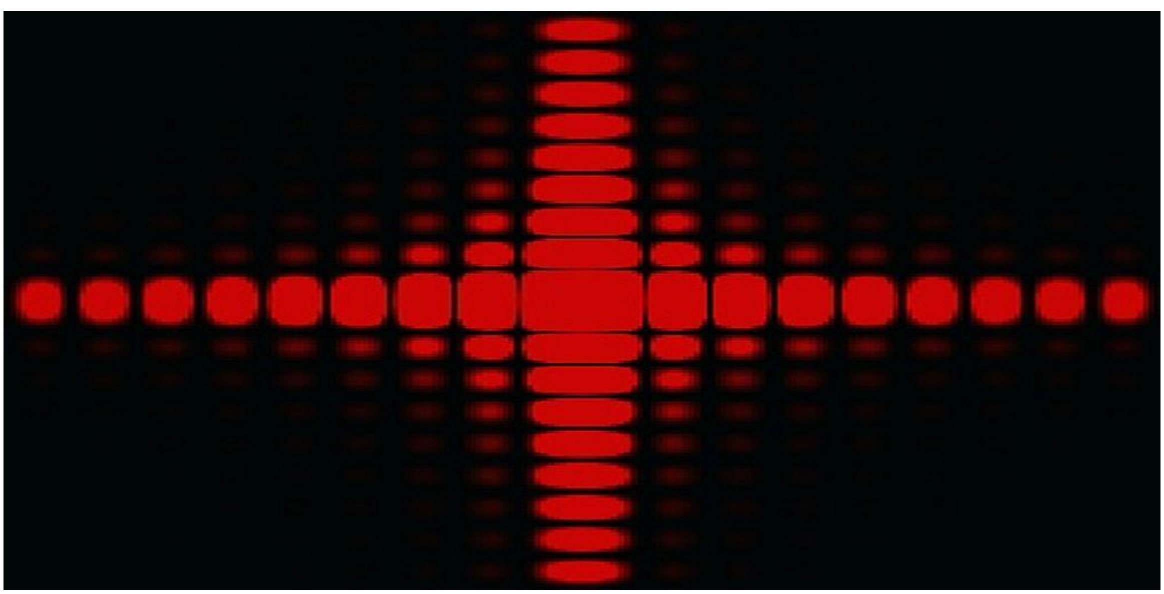
Note that $b = 62.5\ \mu\text{m}$ and $\lambda = 0.6328\ \mu\text{m}$.

Then $\frac{\lambda}{b} = \frac{5 \text{ cm}}{r_o}$ leads to $\frac{0.6328\ \mu\text{m}}{62.5\ \mu\text{m}} = \frac{5 \text{ cm}}{r_o}$.

$$r_o = \frac{62.5\mu\text{m}}{0.6328\ \mu\text{m}} \cdot 5 \text{ cm} = 494 \text{ cm}$$

$$\boxed{r_o = 4.9 \text{ m}}$$

HW T2. Rectangular Aperture. A rectangular aperture is b = 0.30 mm wide and h = 0.10 mm tall. A helium-neon (He-Ne) laser is used to send its 632.8-nm red light through the aperture. The screen on which the diffraction pattern falls is 1 meter away from the aperture. Give the dimensions in mm to three significant figures for the central rectangular bright region that appears on the screen. Take the laser beam large enough so that you can consider the light that reaches the aperture as plane waves.



Wikipedia: Epzcaw. Creative Commons

$$I(\theta) = I(0)\left[\frac{\sin\alpha}{\alpha}\right]^2\left[\frac{\sin\beta}{\beta}\right]^2 \qquad \alpha = \frac{1}{2}kh\sin\phi \qquad \beta = \frac{1}{2}kb\sin\theta$$

We want $\sin\alpha$ and $\sin\beta$ when they are first equal to zero on either side of the center. Therefore,

$\alpha = \pm\pi$ and $\beta = \pm\pi$.

Then $\alpha = \frac{1}{2}kh\sin\phi = \pm\pi$ and $\beta = \frac{1}{2}kb\sin\theta = \pm\pi$.

$$\alpha = \frac{1}{2}\frac{2\pi}{\lambda}h\sin\phi = \pm\pi \qquad \beta = \frac{1}{2}\frac{2\pi}{\lambda}b\sin\theta = \pm\pi$$

$$\sin\phi = \pm\frac{\lambda}{b} = \frac{y}{r_o} \qquad \sin\theta = \pm\frac{\lambda}{b} = \frac{x}{r_o}$$

These relationships are similar to the $\sin\theta = \frac{5\text{ cm}}{r_o}$ of the previous assignment.

Horizontal width: $\sin\theta = \pm\frac{\lambda}{b} = \frac{x}{r_o}$ => $\Delta x = 2\frac{\lambda}{b}r_o$

Vertical width: $\sin\phi = \pm\frac{\lambda}{h} = \frac{y}{r_o}$ => $\Delta y = 2\frac{\lambda}{h}r_o$

$b = 0.3\text{ mm} = 300\ \mu\text{m}$, $h = 0.1\text{ mm} = 100\ \mu\text{m}$ $\lambda = 0.6328\ \mu\text{m}$, $r_o = 1\text{ m}$.

$$\Delta x = 2\frac{\lambda}{b}r_o = 2\frac{0.6328\ \mu\text{m}}{300\ \mu\text{m}}(1000\text{ mm}) = \frac{632.8}{150} = 4.22\text{ mm}$$

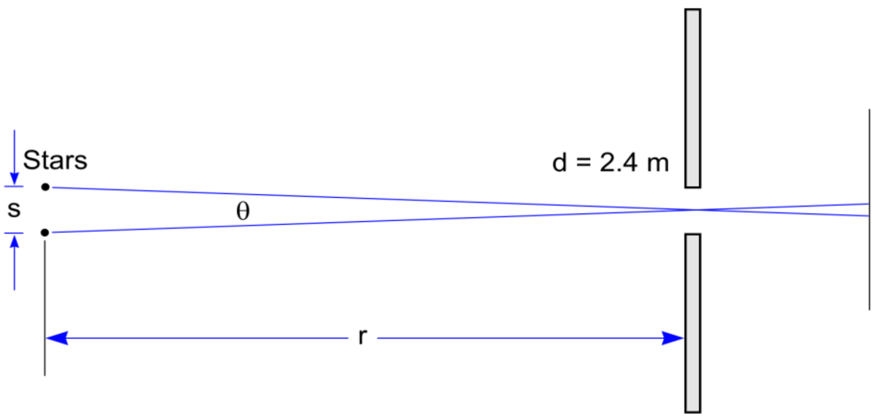
$$\Delta y = 2\frac{\lambda}{h}r_o = 2\frac{0.6328\ \mu\text{m}}{100\ \mu\text{m}}(1000\text{ mm}) = \frac{632.8}{50} = 3\cdot 4.22 = 12.7\text{ mm}$$

$$\boxed{\Delta x = 4.22\text{ mm}} \qquad \boxed{\Delta y = 12.7\text{ mm}}$$

HW T3. Binary Stars. There is a binary star system at a distance of 10 light years from the Earth. A binary star system consists of two stars in orbit about each other. Use the Rayleigh criterion (which is based on the Airy disk),

$$d \sin\theta = 1.22\lambda \,,$$

to determine the minimum distance between the stars in astronomical units (AU) so that the Hubble telescope's 2.40 meter mirror can barely resolve the two stars. Use $\lambda = 550 \text{ nm}$, the wavelength at which human day vision is most sensitive. The steps are outlined below.



a) Use the Rayleigh criterion equation with the Hubble telescope diameter and wavelength 550 nm to find the angle θ. This angle is same angle on each side of the crossing light rays in the figure. You want one light ray to fall at the edge of the Airy disk of the other. Thus, we use the Rayleigh criterion.

b) Then relate the arc length s, to r, and θ. You can approximate the distance between the stars as this arc length.

c) The astronomical unit AU is average distance between the Sun and the Earth. A light year is the distance light travels in one year. You will need to work with these units to finish the calculation.

SOLUTION

You can give numerical answers as you go or you can plug in numbers last like we do below.

We will consider all the data given to be good to three significant figures: the 10 ly, 550 nm.

a) Rayleigh criterion: $d \sin\theta = 1.22\lambda$. For super small angles: $(d)(\theta) = 1.22\lambda$

b) Arc length relation: $s = r\theta$

c) Astronomical unit and light year: $1 \text{ ly} = 6.324 \times 10^4 \text{ AU}$

From (b) $s = r\theta$ and from (a) $\theta = \dfrac{1.22\lambda}{d}$, we get $s = r\dfrac{1.22\lambda}{d}$.

$$s = (10.0 \text{ ly})\frac{1.22(550\times10^{-9} \text{ m})}{2.40 \text{ m}}\frac{6.324\times10^4 \text{ AU}}{1 \text{ ly}} = 0.1768 \text{ AU}$$

$$\boxed{s = 0.177 \text{ AU}}$$

Mercury is 0.4 AU from the Sun so these binary stars are close!