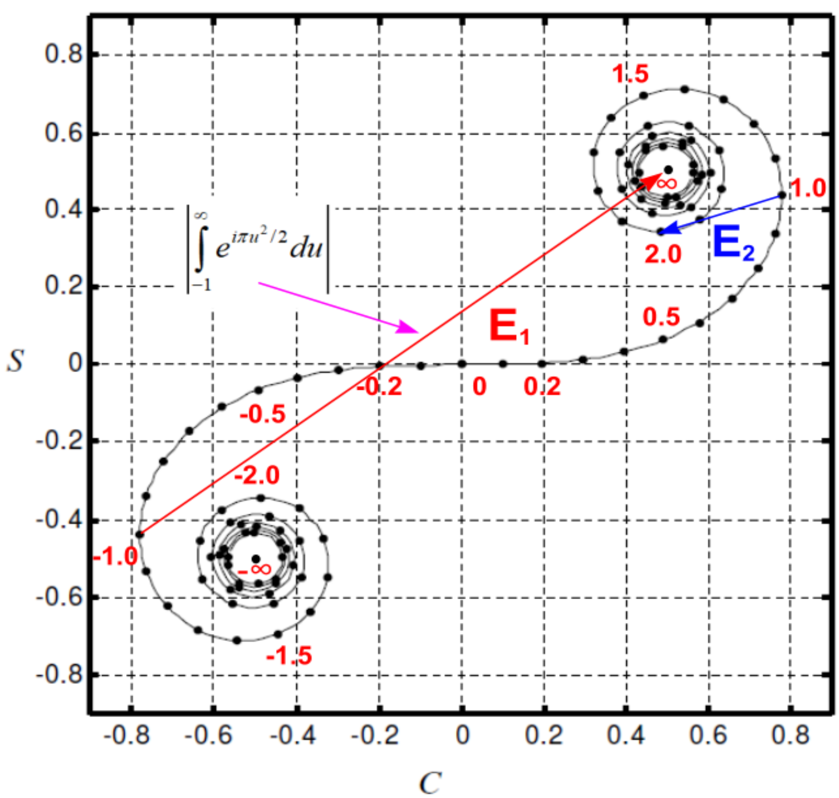
HW V1. Cornu Spiral. Use the Cornu spiral to evaluate the following integrals and the corresponding modulus in each case, showing all steps in the process. Do your best in reading the graph below to report all final numerical answers to two significant figures.

(a) $E_1 = \int_{-1}^{\infty} e^{i\pi u^2/2} du$ and its corresponding modulus $|E_1| = \left| \int_{-1}^{\infty} e^{i\pi u^2/2} du \right|$

(b) $E_2 = \int_{1}^{2} e^{i\pi u^2/2} du$ and its corresponding modulus $|E_2| = \left| \int_{1}^{2} e^{i\pi u^2/2} du \right|$



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SOLUTIONS

$$E_1 = \int_{-1}^{\infty} e^{i\pi u^2/2} du$$
$$= [C(\infty) - C(-1)]$$
$$+ i[S(\infty) - S(-1)]$$

Your numbers below might be a little different due to the element of subjectivity in reading the rather crude graph.

$$E_1 = [0.50 - (-0.78)]$$
$$+ i[0.50 - (-0.42)]$$

$$E_1 = 1.28 + 0.92i \quad => \quad |E_1| = \sqrt{1.28^2 + 0.92^2} = 1.576 \quad => \quad \boxed{|E_1| = 1.6}$$

$$E_2 = \int_{1}^{2} e^{i\pi u^2/2} du = \left[C(u) + iS(u)\right]_1^2 = \left[C(2) - C(1)\right] + i\left[S(2) - S(1)\right]$$

$$E_2 = \left[0.48 - 0.78\right] + i\left[0.35 - 0.43\right] = -0.30 - 0.10i$$

$$|E_2| = \sqrt{0.30^2 + 0.10^2} = 0.316 \quad => \quad \boxed{|E_2| = 0.32}$$

HW V2. Asymptotic Form for the Fresnel Integrals. In this problem you will work with the Fresnel integrals with the purely mathematical variables below. These variables do not have any physics relevance in this context, e.g., x is not distance, t is not time, and u is not our u from class, etc. With these variables, the Fresnel integrals can be expressed as

$$C(x) = \int_0^x \cos(\frac{\pi \cdot t^2}{2})dt \quad \text{and} \quad S(x) = \int_0^x \sin(\frac{\pi \cdot t^2}{2})dt.$$

Note that to play it safe in a strict mathematical sense, we are using a different variable "t" for the integration variable and another for the upper limit "x." You will be led through to derive the asymptotic form for the S(x) Fresnel integral. The form is valid for very large values of x. Write S(x) as

$$S(x) = \int_0^x \sin(\frac{\pi \cdot t^2}{2})dt = \int_0^\infty \sin(\frac{\pi \cdot t^2}{2})dt - \int_x^\infty \sin(\frac{\pi \cdot t^2}{2})dt.$$

Evaluate the first integral the usual way by consulting the Cornu spiral. For the second integral, make a substitution of variable by setting $u = \pi t^2 / 2$ and show that with this change of variables

$$\int_x^\infty \sin(\frac{\pi \cdot t^2}{2})dt = \frac{1}{\sqrt{2\pi}} \int_y^\infty \frac{\sin u}{\sqrt{u}} du \text{ , where } y = \frac{\pi x^2}{2}$$

Now use integration by parts twice to show that

$$\int_y^\infty \frac{\sin u}{\sqrt{u}} du = \frac{\cos y}{y^{1/2}} + \frac{\sin y}{2y^{3/2}} - \frac{3}{4}\int_y^\infty \frac{\sin u}{u^{5/2}} du.$$

Since x is large $y = \pi x^2/2$ is super large. These super large denominators mean we can just keep the term with $\frac{1}{y^{1/2}}$ and neglect the higher powers such as $\frac{1}{y^{3/2}}$ and $\frac{1}{y^{5/2}}$, the latter which would emerge if you did a third integration by parts. Pull everything together to show that the asymptotic form is

$$S(x) \approx \frac{1}{2} - \frac{1}{\pi x}\cos(\frac{\pi x^2}{2}).$$

For full credit you must do the integration by parts by hand and show all steps of the integration.

You need not do the C(x) case since the steps are similar and you can practically guess the answer. Compare this cosine answer below to the sine one you derived for this assignment. Why the plus sign for the C(x) case?

$$C(x) \approx \frac{1}{2} + \frac{1}{\pi x}\sin(\frac{\pi x^2}{2}).$$

SOLUTIONS

$$S(x) = \int_0^x \sin(\frac{\pi \cdot t^2}{2})dt = \int_0^\infty \sin(\frac{\pi \cdot t^2}{2})dt - \int_x^\infty \sin(\frac{\pi \cdot t^2}{2})dt$$

$$S(x) = I_1 + I_2$$

$$I_1 = \int_0^\infty \sin(\frac{\pi \cdot t^2}{2})dt = S(x)\Big|_0^\infty = S(\infty) - S(0) = \frac{1}{2} - 0 = \frac{1}{2}.$$

That takes care of the first integral. The second integral is

$$I_2 = -\int_x^\infty \sin(\frac{\pi \cdot t^2}{2})dt.$$

Change variables: $u = \pi t^2 / 2$.

$$du = \pi t dt \quad \Rightarrow \quad dt = \frac{du}{\pi t} \quad \Rightarrow \quad dt = \frac{du}{\pi t}$$

Also we have $t = \sqrt{\frac{2u}{\pi}}$ and $dt = \frac{du}{\pi\sqrt{\frac{2u}{\pi}}} = \frac{du}{\sqrt{2\pi u}}$.

$$x \le t \le \infty$$

$$\frac{\pi x^2}{2} \le u \le \infty$$

Our integral is then $I_2 = -\int_y^\infty \sin(u)\frac{du}{\sqrt{2\pi u}}$ with $y = \frac{\pi x^2}{2}$.

$$I_2 = -\frac{1}{\sqrt{2\pi}} \int_y^\infty \frac{\sin(u)}{\sqrt{u}}\, du$$, which we can write as $$-\sqrt{2\pi} I_2 = \int_y^\infty \frac{\sin(u)}{\sqrt{u}}\, du$$.

At this point, using integration by parts, we are asked to show

$$\int_y^\infty \frac{\sin u}{\sqrt{u}}\, du = \frac{\cos y}{y^{1/2}} + \frac{\sin y}{2y^{3/2}} - \frac{3}{4}\int_y^\infty \frac{\sin u}{u^{5/2}}\, du$$.

Integration by parts is based on

$$d(fg) = f dg + g df$$ and $$\int d(fg) = \int f dg + \int g df$$

$$\int f dg = \int d(fg) - \int g df$$

The first integration by parts with $\int_y^\infty \frac{\sin u}{\sqrt{u}}\, du$ will be done now.

Since we know $\cos u$ appears in the answer, we choose

$$f = \frac{1}{\sqrt{u}}$$ and $$dg = (\sin u) du$$.

Then $g = -\cos u$.

$$\int f dg = \int d(fg) - \int g df \rightarrow -\frac{\cos u}{\sqrt{u}}\Bigg|_y^\infty - \int_y^\infty (-\cos u) d(\frac{1}{\sqrt{u}})$$

$$\int_y^\infty \frac{\sin u}{\sqrt{u}}\, du = \frac{\cos y}{\sqrt{y}} + \int_y^\infty (\cos u) d(\frac{1}{\sqrt{u}})$$

$$\int_y^\infty \frac{\sin u}{\sqrt{u}}\, du = \frac{\cos y}{\sqrt{y}} - \frac{1}{2}\int_y^\infty \frac{\cos u}{u^{3/2}}\, du$$

Summary so far:

$$S(x) = I_1 + I_2 \text{ with}$$

$$I_1 = \frac{1}{2} \quad \text{and} \quad -\sqrt{2\pi} I_2 = \int_y^\infty \frac{\sin(u)}{\sqrt{u}} du.$$

$$\int_y^\infty \frac{\sin(u)}{\sqrt{u}} du = \frac{\cos y}{\sqrt{y}} - \frac{1}{2}\int_y^\infty \frac{\cos u}{u^{3/2}} du$$

The second integration by parts is on

$$\int_y^\infty \frac{\cos u}{u^{3/2}} du$$

$$f = \frac{1}{u^{3/2}} \text{ and } dg = (\cos u) du.$$

Then $g = \sin u$.

$$\int fdg = \int d(fg) - \int gdf \rightarrow \left.\frac{\sin u}{u^{3/2}}\right|_y^\infty - \int_y^\infty (\sin u) d(\frac{1}{u^{3/2}})$$

$$\int_y^\infty \frac{\cos u}{u^{3/2}} du = -\frac{\sin y}{y^{3/2}} + \frac{3}{2}\int_y^\infty \frac{\sin u}{u^{5/2}} du$$

Put everything together.

$$\int_y^\infty \frac{\sin(u)}{\sqrt{u}} du = \frac{\cos y}{\sqrt{y}} - \frac{1}{2}\int_y^\infty \frac{\cos u}{u^{3/2}} du$$

$$\int_y^\infty \frac{\sin(u)}{\sqrt{u}} du = \frac{\cos y}{\sqrt{y}} - \frac{1}{2}\left[-\frac{\sin y}{y^{3/2}} + \frac{3}{2}\int_y^\infty \frac{\sin u}{u^{5/2}} du\right]$$

$$\int_y^\infty \frac{\sin u}{\sqrt{u}}\,du = \frac{\cos y}{y^{1/2}} + \frac{\sin y}{2y^{3/2}} - \frac{3}{4}\int_y^\infty \frac{\sin u}{u^{5/2}}\,du$$

Final Summary: $S(x) = I_1 + I_2$ with

$$I_1 = \frac{1}{2} \quad \text{and} \quad -\sqrt{2\pi}\,I_2 = \int_y^\infty \frac{\sin(u)}{\sqrt{u}}\,du.$$

$$\int_y^\infty \frac{\sin u}{\sqrt{u}}\,du = \frac{\cos y}{y^{1/2}} + \frac{\sin y}{2y^{3/2}} - \frac{3}{4}\int_y^\infty \frac{\sin u}{u^{5/2}}\,du$$

Keeping the first term. $\int_y^\infty \frac{\sin u}{\sqrt{u}}\,du \approx \frac{\cos y}{y^{1/2}}$

$$I_2 = -\frac{1}{\sqrt{2\pi}}\int_y^\infty \frac{\sin(u)}{\sqrt{u}}\,du \approx -\frac{1}{\sqrt{2\pi}}\frac{\cos y}{y^{1/2}}$$

$$S(x) = I_1 + I_2 \quad \Rightarrow \quad S(x) \approx \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\frac{\cos y}{y^{1/2}}$$

We now remember $y = \frac{\pi x^2}{2}$. Therefore, $y^{1/2} = \sqrt{\frac{\pi}{2}}x$ and

$$S(x) \approx \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\frac{1}{x}\cos\frac{\pi x^2}{2} \quad \Rightarrow \quad \boxed{S(x) \approx \frac{1}{2} - \frac{1}{\pi x}\cos\frac{\pi x^2}{2}}$$

$$C(x) \approx \frac{1}{2} + \frac{1}{\pi x}\sin(\frac{\pi x^2}{2})$$

C(x) will have a flipped sign with sine due to the integration by parts on the cosine versus the sine.

$dg = (\sin u)du$ => $g = -\cos u$, while $dg = (\cos u)du$ => $g = \sin u$.