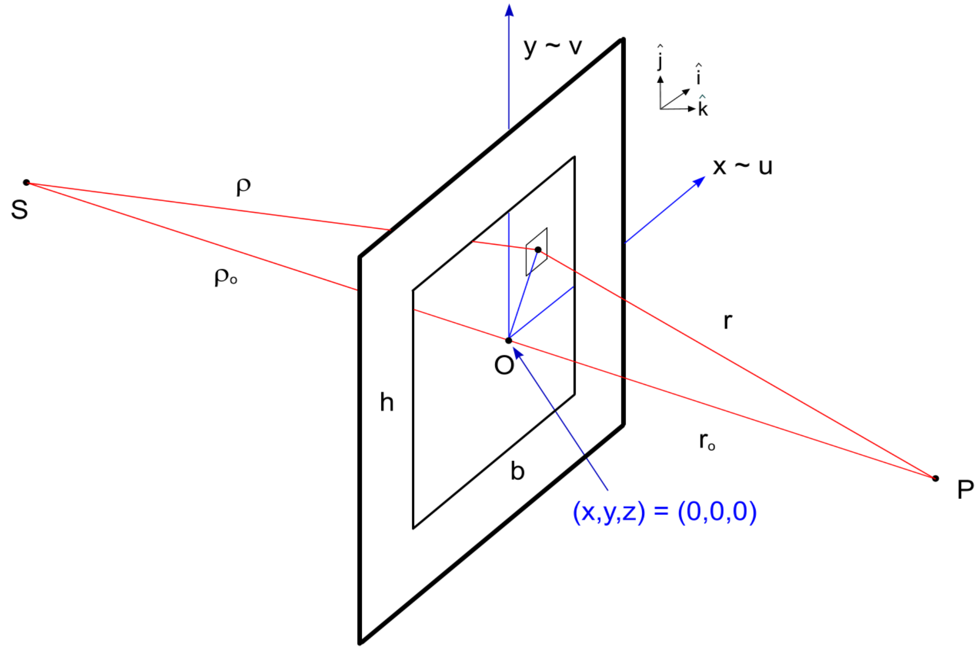
W0. Summary from Last Class. For a rectangular aperture we have the figure below, where the light source is at point S and the observed diffraction at point P.



$$E_p = C\int_{x_1}^{x_2}\int_{y_1}^{y_2} e^{ik\frac{(x^2+y^2)}{2}\left(\frac{\rho_o+r_o}{\rho_o r_o}\right)} dxdy$$

With the variables: $u = x\left[\dfrac{2(\rho_o + r_o)}{\lambda \rho_o r_o}\right]^{1/2}$ and $v = y\left[\dfrac{2(\rho_o + r_o)}{\lambda \rho_o r_o}\right]^{1/2}$

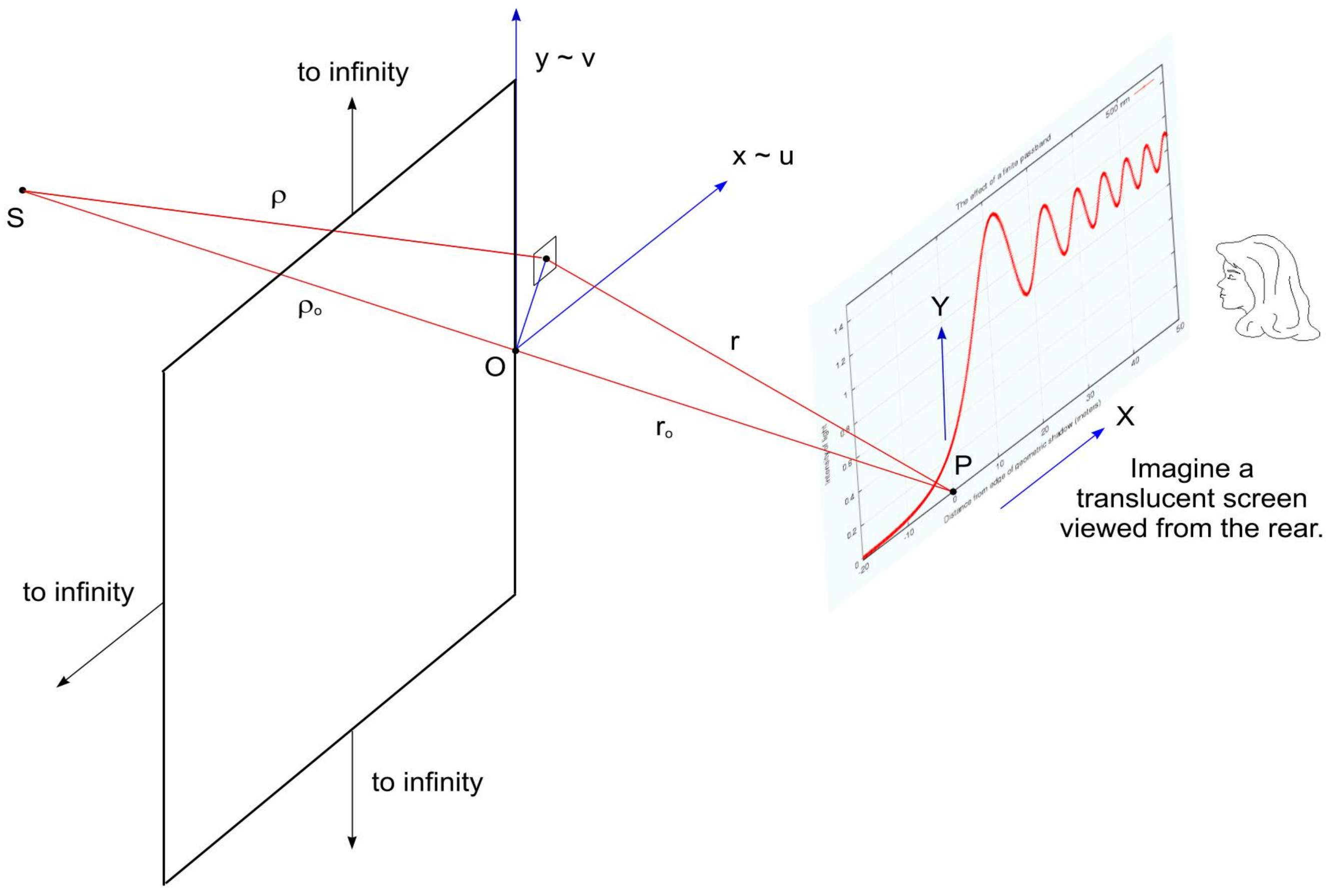
$$E_p = \frac{A_o}{2}\int_{u_1}^{u_2} e^{i\pi u^2/2}du\int_{v_1}^{v_2} e^{i\pi v^2/2}dv$$ since it is easier to work with u and v.

If we enlarge the aperture to infinity, i.e., h = b = ∞, we have the unobstructed wave.

$$E_p = \frac{A_o}{2}\int_{-\infty}^{\infty} e^{i\pi u^2/2}du\int_{-\infty}^{\infty} e^{i\pi v^2/2}dv = \frac{A_o}{2}(1+i)(1+i)$$

$$\left|E_p\right| = \frac{A_o}{2}\left|(1+i)(1+i)\right| = \frac{A_o}{2}\left|(1+i)\right|\left|(1+i)\right| = \frac{A_o}{2}\sqrt{2}\sqrt{2} = A_o$$

W1. Fresnel Edge. Here we analyze the edge of a vertical wall that is infinite along the vertical and extends to infinity along the negative horizontal axis. The diffraction pattern falls on a translucent screen that is viewed from the rear side.

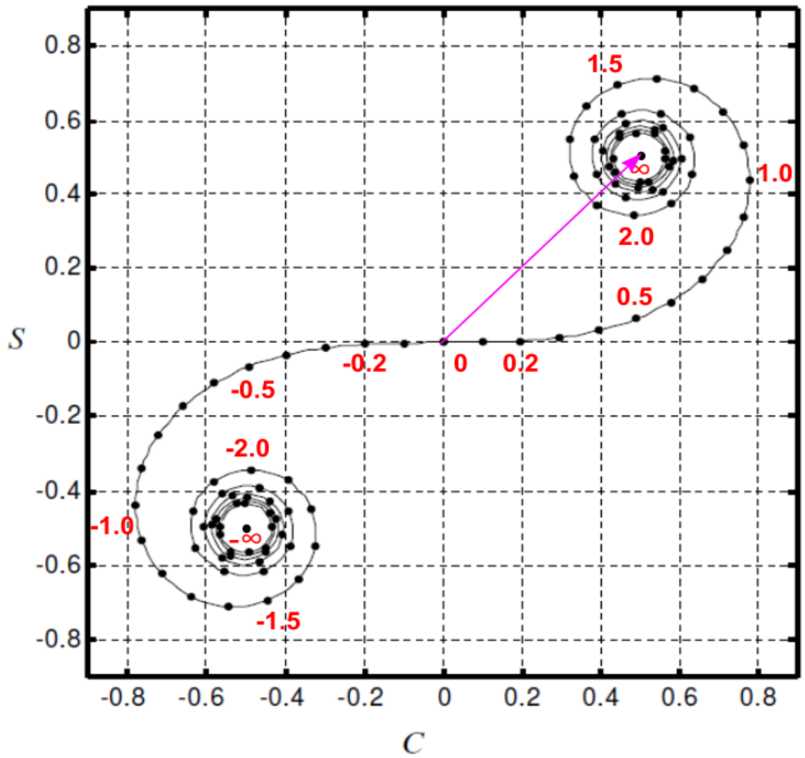


At point P we want to integrate over the "aperture" region, a half plane.

$$E_p = \frac{A_o}{2}\int_{u_1}^{u_2} e^{i\pi u^2/2}du \int_{v_1}^{v_2} e^{i\pi v^2/2}dv \quad \Rightarrow \quad E_p = \frac{A_o}{2}\int_{0}^{\infty} e^{i\pi u^2/2}du \int_{-\infty}^{\infty} e^{i\pi v^2/2}dv$$

Consult the Cornu spiral below.

The integrals are $\int_{0}^{\infty} e^{i\pi u^2/2}du = \frac{1+i}{2}$ and $\int_{-\infty}^{\infty} e^{i\pi v^2/2}dv = 1+i$.



$$E_p = \frac{A_o}{2}\frac{(1+i)}{2}(1+i)$$

$$|E_p| = \frac{A_o}{2}\frac{\sqrt{2}}{2}\sqrt{2}$$

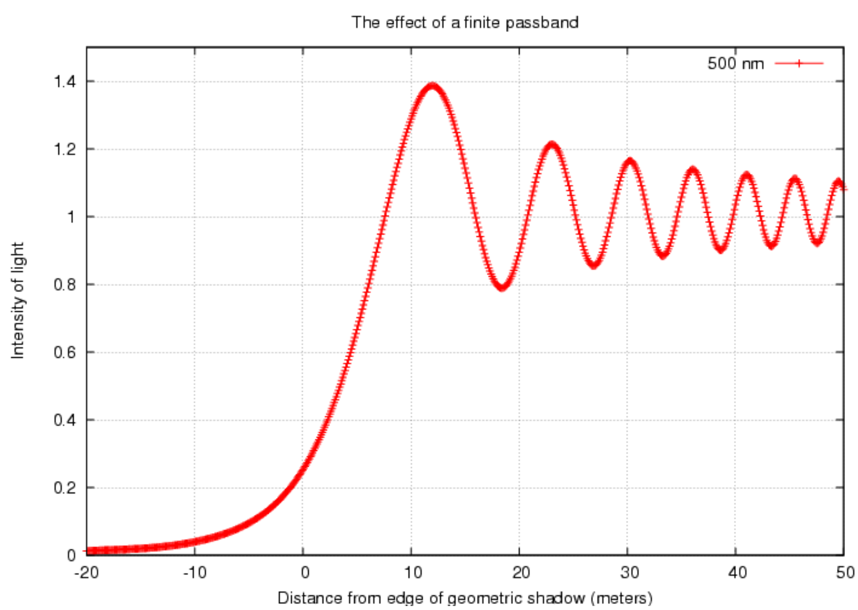
$$|E_p| = \frac{A_o}{2}$$

$|E_p|$ is 1/2 the amplitude of the unobstructed case.

Therefore, the irradiance is 1/4 the strength of the unobstructed case. All the detailed steps are written out below for mastery.

$$I_o = \frac{1}{2}A_o^2 \quad \text{and} \quad I_p = \frac{1}{2}|E_p|^2 = \frac{1}{2}\left[\frac{A_o}{2}\right]^2 = \frac{1}{2}\frac{A_o^2}{4} = \frac{1}{4}I_o$$

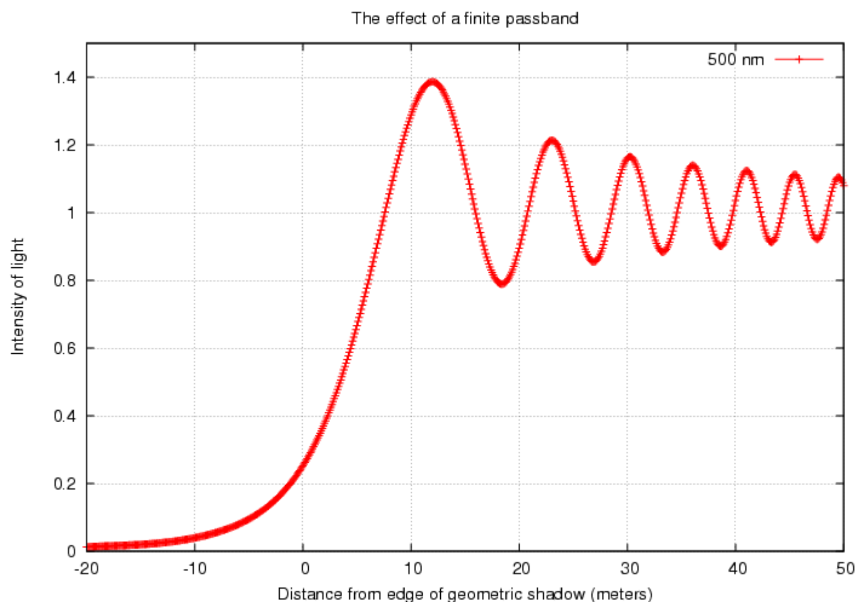
See the place in the graph below where the distance is 0. The intensity compared to the unobstructed beam is shown in the figure to be at 0.25, courtesy Michael Richmond.



W2. An Astrophysics Example: Fresnel Fringe Effects During Lunar Occultation

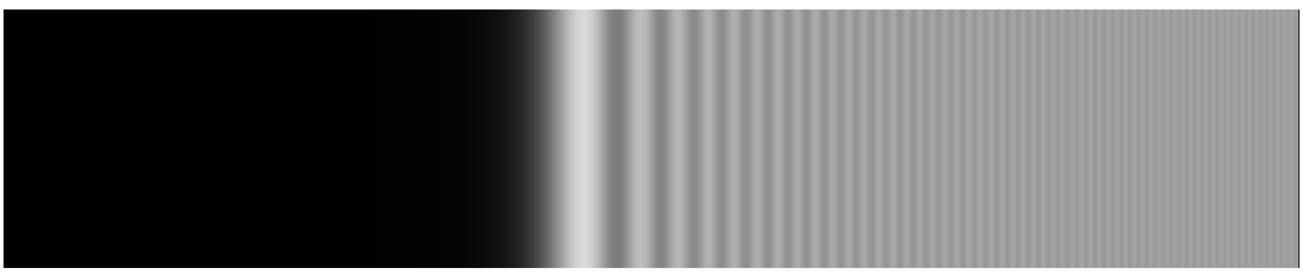
This figure described the relative irradiance compared to the unobstructed beam. It goes with the actual screen appearance shown below. It applies to all such large obstacles, but in this case we are applying it to the edge of the Moon lined up with a distance star. Finite passband means restricted wavelengths, in this case 500 nm light.

Fresnel Diffraction Irradiance Graph Due to an Obstacle Wall



Courtesy Prof. Michael Richmond, Rochester Institute of Technology. Creative Commons

Simulated Fresnel Diffraction Irradiance on the Screen Due to an Obstacle Wall



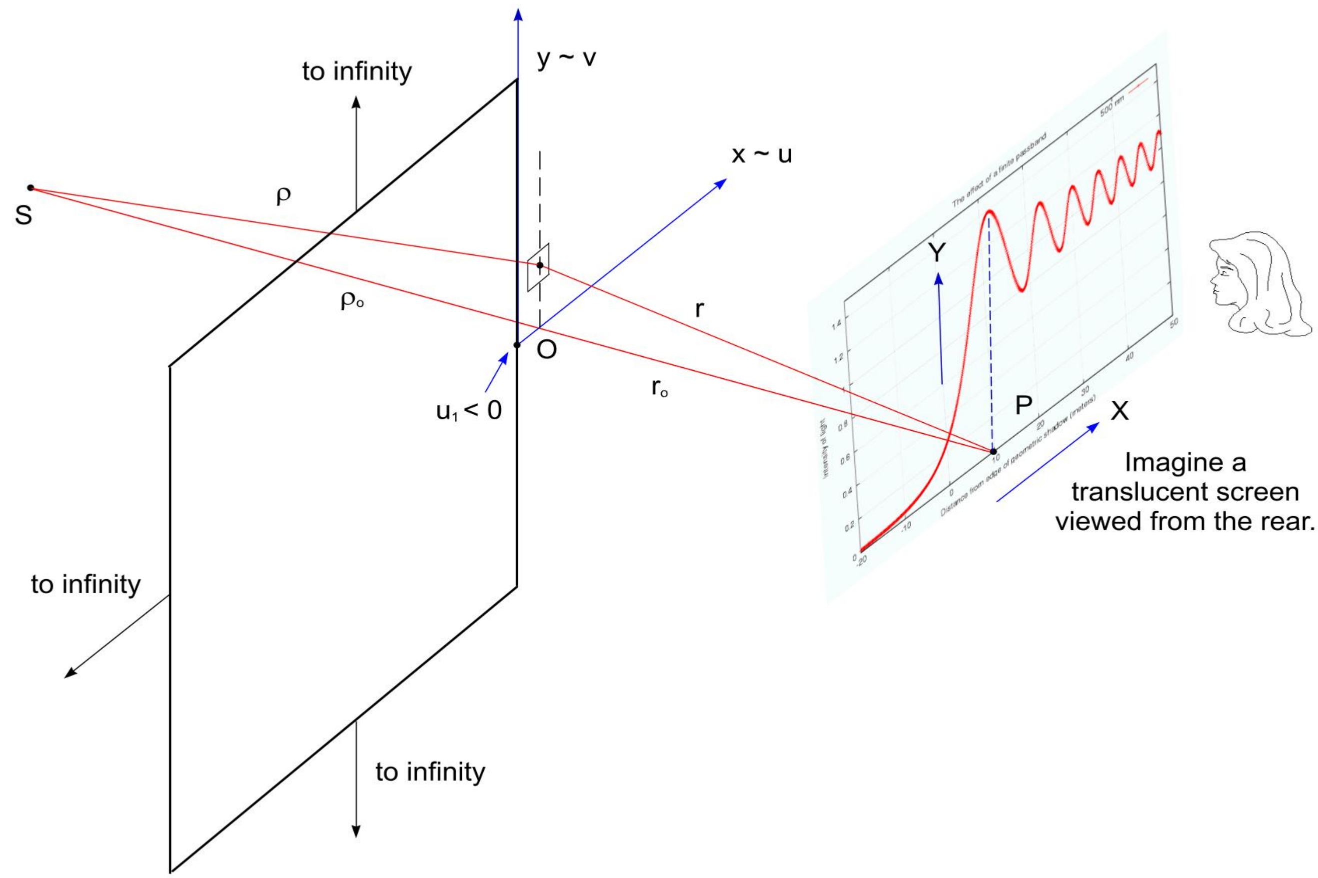
Courtesy Prof. Michael Richmond, Rochester Institute of Technology. Creative Commons
with a Credit Line to Prof. Roger L. Easton, Jr. of RIT's Center for Imaging Sciences.

Prof. Richmond is analyzing light from a star passing at the limb of the Moon.

The Moon can be taken to be the obstacle wall in the vicinity of the light path from the star.

For this specific application, distances X from point P are given in meters.
We have calculated the irradiance for point P at distance zero.

Next we would like to get the first maximum of relative irradiance. The point P is now shifted.



For our new point P the u integration starts at some $u_1 < 0$.

$$E_p = \frac{A_o}{2} \int_{u_1}^{\infty} e^{i\pi u^2/2} du \int_{-\infty}^{\infty} e^{i\pi v^2/2} dv \quad => \quad E_p = \frac{A_o}{2} \int_{u_1}^{\infty} e^{i\pi u^2/2} du (1+i)$$

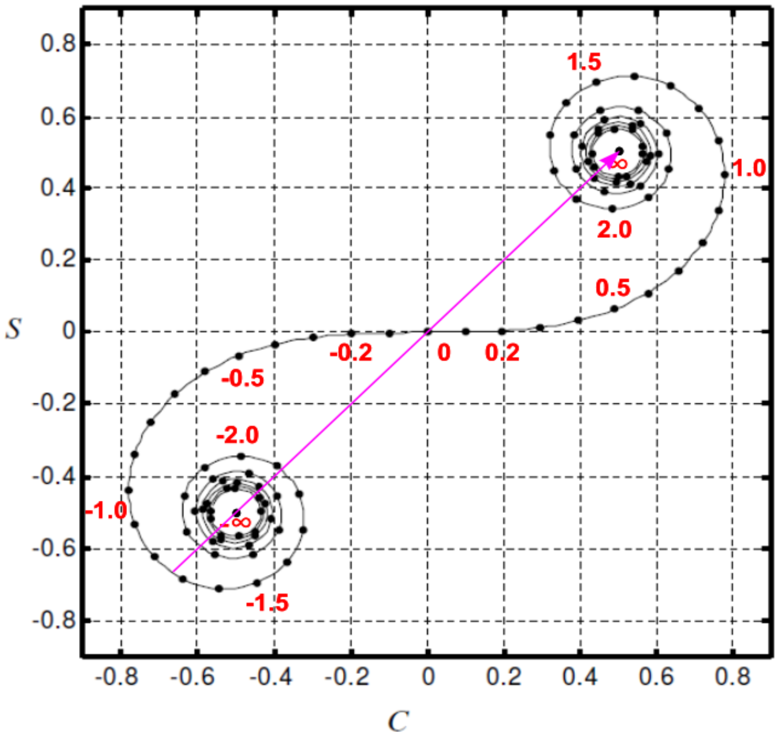
But what is u_1? What shall will pick for u_1?
No problem.

We consult the Cornu spiral and search for the u < 0 that will give the longest line for

$$\int_{u_1}^{\infty} e^{i\pi u^2/2} du \; .$$

Or, see the Fresnel integral table on the next page.

u	C(u)	S(u)		**u**	**C(u)**	**S(u)**
-5.0	-0.5636	-0.4992		0.0	0.0000	0.0000
-4.9	-0.5002	-0.4351		0.1	0.1000	0.0005
-4.8	-0.4338	-0.4968		0.2	0.1999	0.0042
-4.7	-0.4914	-0.5671		0.3	0.2994	0.0141
-4.6	-0.5672	-0.5162		0.4	0.3975	0.0334
-4.5	-0.5260	-0.4343		0.5	0.4923	0.0647
-4.4	-0.4383	-0.4623		0.6	0.5811	0.1105
-4.3	-0.4494	-0.5540		0.7	0.6597	0.1721
-4.2	-0.5417	-0.5632		0.8	0.7228	0.2493
-4.1	-0.5737	-0.4758		0.9	0.7648	0.3398
-4.0	-0.4984	-0.4205		1.0	0.7799	0.4383
-3.9	-0.4223	-0.4752		1.1	0.7638	0.5365
-3.8	-0.4481	-0.5656		1.2	0.7154	0.6234
-3.7	-0.5419	-0.5750		1.3	0.6386	0.6863
-3.6	-0.5880	-0.4923		1.4	0.5431	0.7135
-3.5	-0.5326	-0.4152		1.5	0.4453	0.6975
-3.4	-0.4385	-0.4296		1.6	0.3655	0.6389
-3.3	-0.4057	-0.5193		1.7	0.3238	0.5492
-3.2	-0.4663	-0.5933		1.8	0.3336	0.4509
-3.1	-0.5616	-0.5818		1.9	0.3945	0.3733
-3.0	-0.6057	-0.4963		2.0	0.4883	0.3434
-2.9	-0.5624	-0.4101		2.1	0.5816	0.3743
-2.8	-0.4675	-0.3915		2.2	0.6363	0.4557
-2.7	-0.3925	-0.4529		2.3	0.6266	0.5532
-2.6	-0.3889	-0.5500		2.4	0.5550	0.6197
-2.5	-0.4574	-0.6192		2.5	0.4574	0.6192
-2.4	-0.5550	-0.6197		2.6	0.3889	0.5500
-2.3	-0.6266	-0.5532		2.7	0.3925	0.4529
-2.2	-0.6363	-0.4557		2.8	0.4675	0.3915
-2.1	-0.5816	-0.3743		2.9	0.5624	0.4101
-2.0	-0.4883	-0.3434		3.0	0.6057	0.4963
-1.9	-0.3945	-0.3733		3.1	0.5616	0.5818
-1.8	-0.3336	-0.4509		3.2	0.4663	0.5933
-1.7	-0.3238	-0.5492		3.3	0.4057	0.5193
-1.6	-0.3655	-0.6389		3.4	0.4385	0.4296
-1.5	-0.4453	-0.6975		3.5	0.5326	0.4152
-1.4	-0.5431	-0.7135		3.6	0.5880	0.4923
-1.3	-0.6386	-0.6863		3.7	0.5419	0.5750
-1.2	-0.7154	-0.6234		3.8	0.4481	0.5656
-1.1	-0.7638	-0.5365		3.9	0.4223	0.4752
-1.0	-0.7799	-0.4383		4.0	0.4984	0.4205
-0.9	-0.7648	-0.3398		4.1	0.5737	0.4758
-0.8	-0.7228	-0.2493		4.2	0.5417	0.5632
-0.7	-0.6597	-0.1721		4.3	0.4494	0.5540
-0.6	-0.5811	-0.1105		4.4	0.4383	0.4623
-0.5	-0.4923	-0.0647		4.5	0.5260	0.4343
-0.4	-0.3975	-0.0334		4.6	0.5672	0.5162
-0.3	-0.2994	-0.0141		4.7	0.4914	0.5671
-0.2	-0.1999	-0.0042		4.8	0.4338	0.4968
-0.1	-0.1000	-0.0005		4.9	0.5002	0.4351
0.0	0.0000	0.0000		5.0	0.5636	0.4992



The longest line from a negative u to infinity is shown below, a u-value between -1 and -1.5 We really don't care what it is at this point. Instead, we want the C and S coordinates to do the integral. We recognize immediately that the irradiance will beat the unobstructed source since the line is longer than the one from -∞ to +∞.

$$\int_{u_1}^{\infty} e^{i\pi u^2/2} du$$

$$= [C(\infty) - C(u_1)]$$

$$+ i[S(\infty) - S(u_1)]$$

From the graph roughly $C(u_1) = S(u_1) = -0.7$, and note that they should be equal.

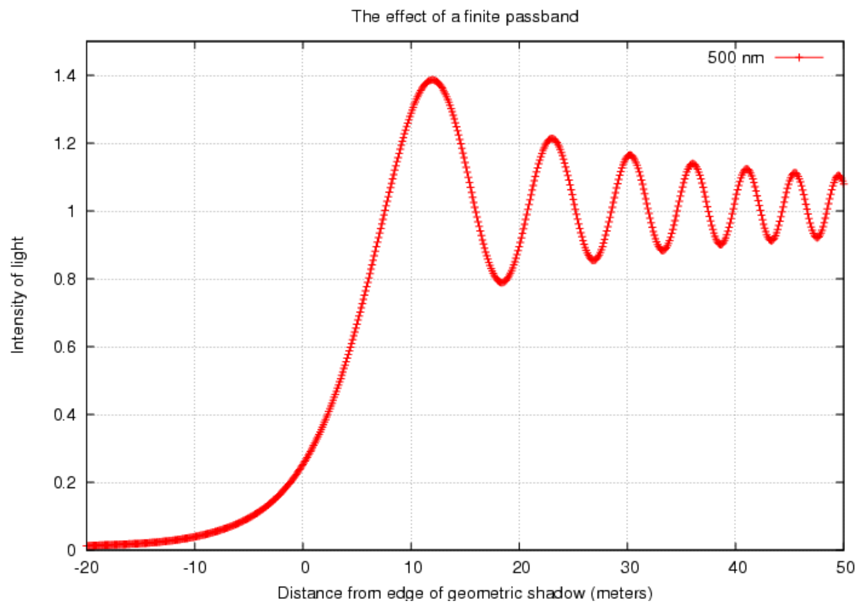
$$\int_{u_1}^{\infty} e^{i\pi u^2/2} du = [C(\infty) - C(u_1)] + i[S(\infty) - S(u_1)]$$

$$\int_{u_1}^{\infty} e^{i\pi u^2/2} du = [0.5 - (-0.7)] + i[0.5 - (-0.7)] = 1.2 + 1.2i = 1.2(1+i)$$

Then $E_p = \dfrac{A_o}{2} \displaystyle\int_{u_1}^{\infty} e^{i\pi u^2/2} du (1+i) = \dfrac{A_o}{2} [(1.2)(1+i)](1+i)$

$$|E_p|^2 = \frac{A_o^2}{4} (1.2)^2 (2)(2) = A_o^2 (1.2)^2 = 1.4 A_o^2$$

Check out the figure on the next page. It is 1.4.



What about the first minimum, where the relative irradiance drops to 0.8.

We move to a new u_1 that now is farther along the spiral.

$$E_p = \frac{A_o}{2} \int_{u_1}^{\infty} e^{i\pi u^2/2} du \int_{-\infty}^{\infty} e^{i\pi v^2/2} dv$$

Since that second integral

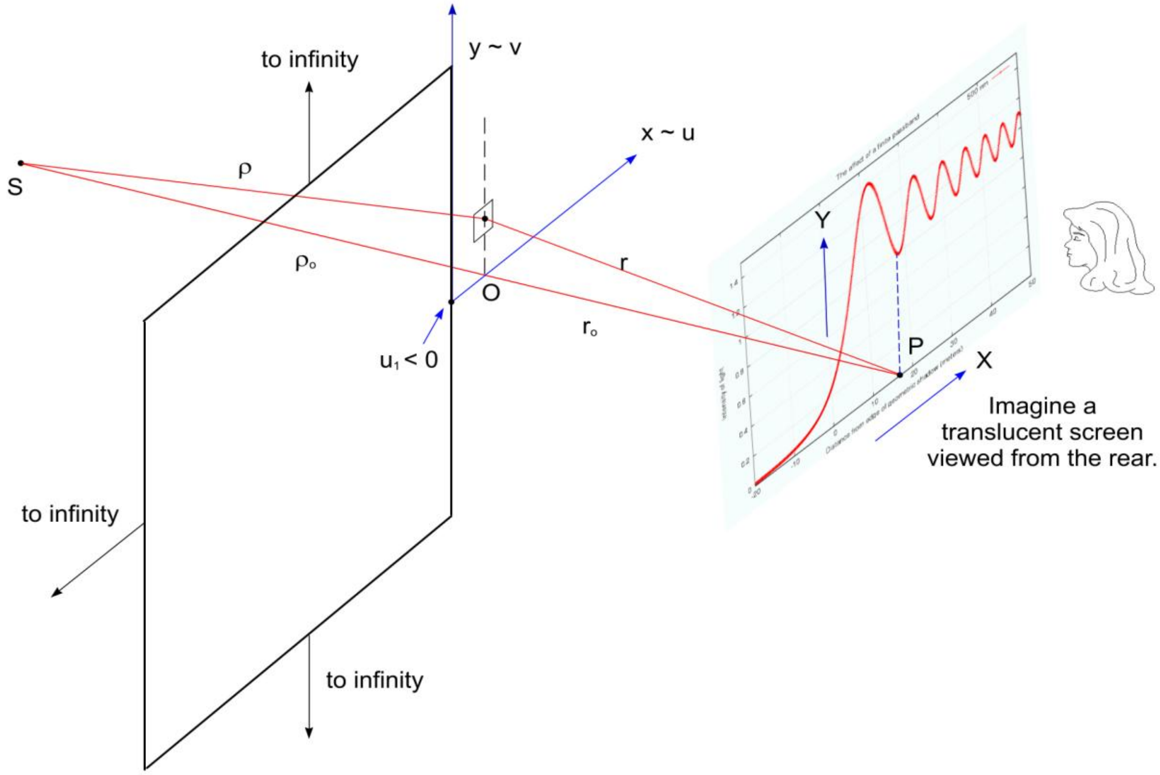
$$\int_{-\infty}^{\infty} e^{i\pi v^2/2} dv = (1+i) \quad \text{with} \quad \left| \int_{-\infty}^{\infty} e^{i\pi v^2/2} dv \right| = |(1+i)| = \sqrt{2},$$

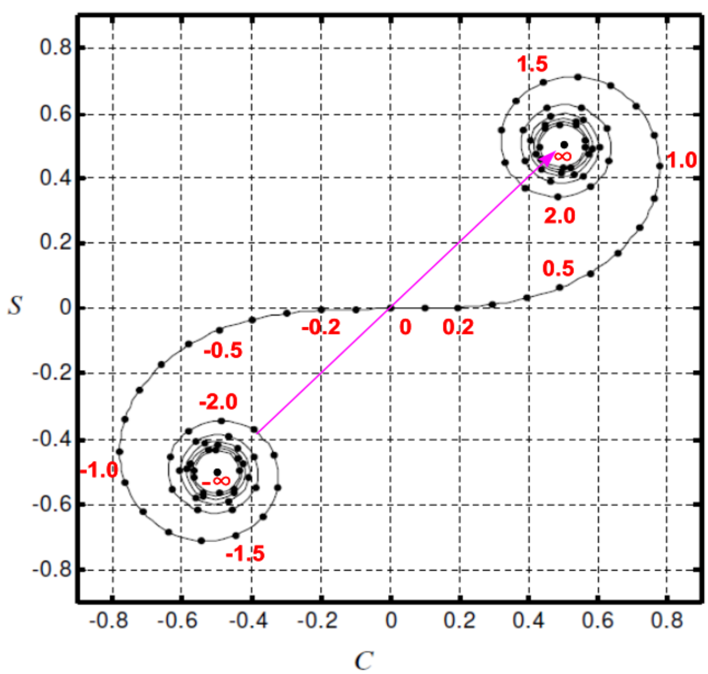
is always the same for our wall, let's write

$$E_p = \frac{A_o}{2} \int_{u_1}^{\infty} e^{i\pi u^2/2} du \sqrt{2}$$

$$\boxed{E_p = \frac{A_o}{\sqrt{2}} \int_{u_1}^{\infty} e^{i\pi u^2/2} du}$$

For the first minimum. Point P is farther along the X axis and u_1 is more to the left of O.





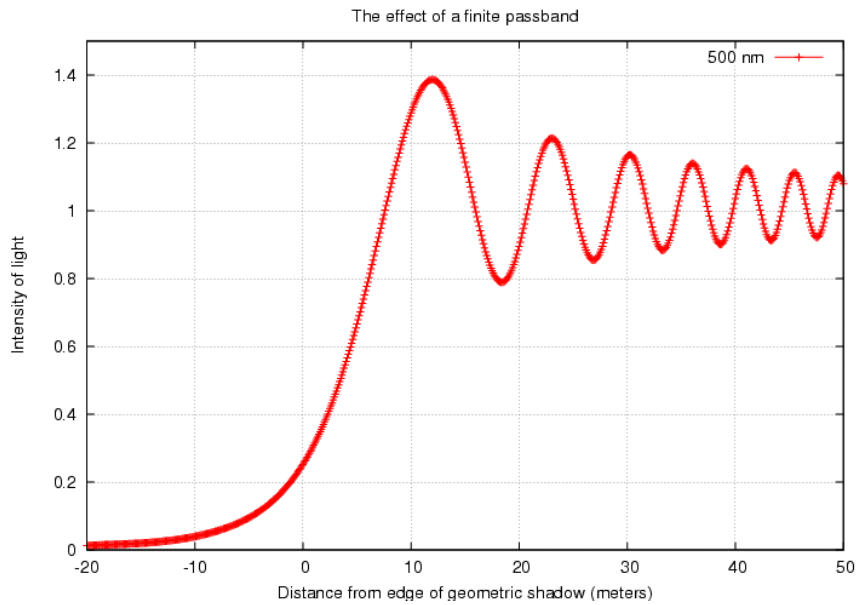
See the Cornu spiral for the first minimum where (C,S) = (-0.4,-0.4) to one significant figure. The length of the line is now

$$\sqrt{2}(0.4+0.5)=0.9\sqrt{2}\,.$$

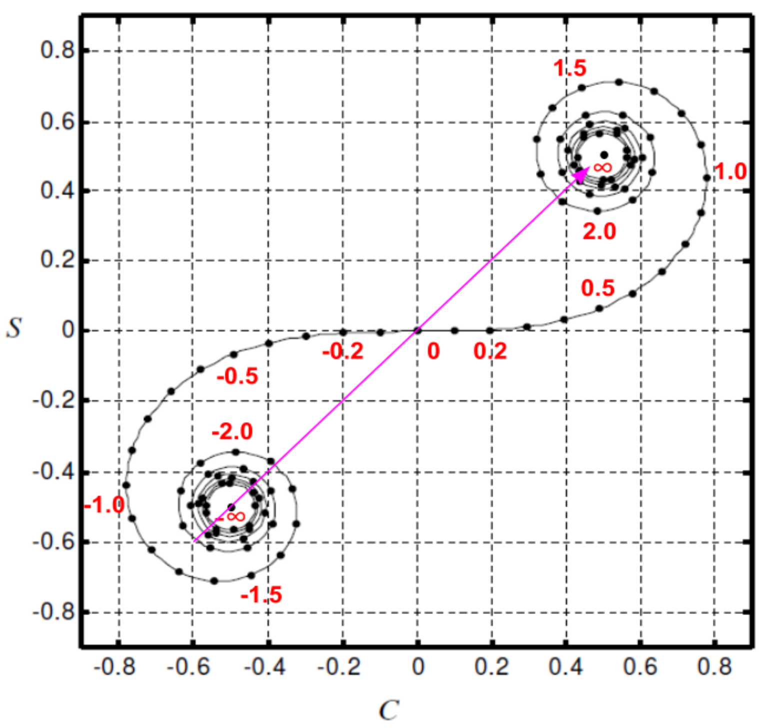
$$E_p=\frac{A_o}{\sqrt{2}}\int_{u_1}^{\infty}e^{i\pi u^2/2}du$$

$$\left|E_p\right|=\frac{A_o}{\sqrt{2}}0.9\sqrt{2}=0.9A_o$$

The irradiance is $I_p=\frac{1}{2}\left|E_p\right|^2=0.9^2\,\frac{A_o^2}{2}=0.8I_o$. There is our 0.8.



The Next max is 1.2.



From the Cornu spiral for the second maximum, (C,S) = (-0.6,-0.6). The length of the line is now

$$\sqrt{2}(0.6+0.5)=1.1\sqrt{2}\,.$$

$$E_p=\frac{A_o}{\sqrt{2}}\int_{u_1}^{\infty}e^{i\pi u^2/2}du$$

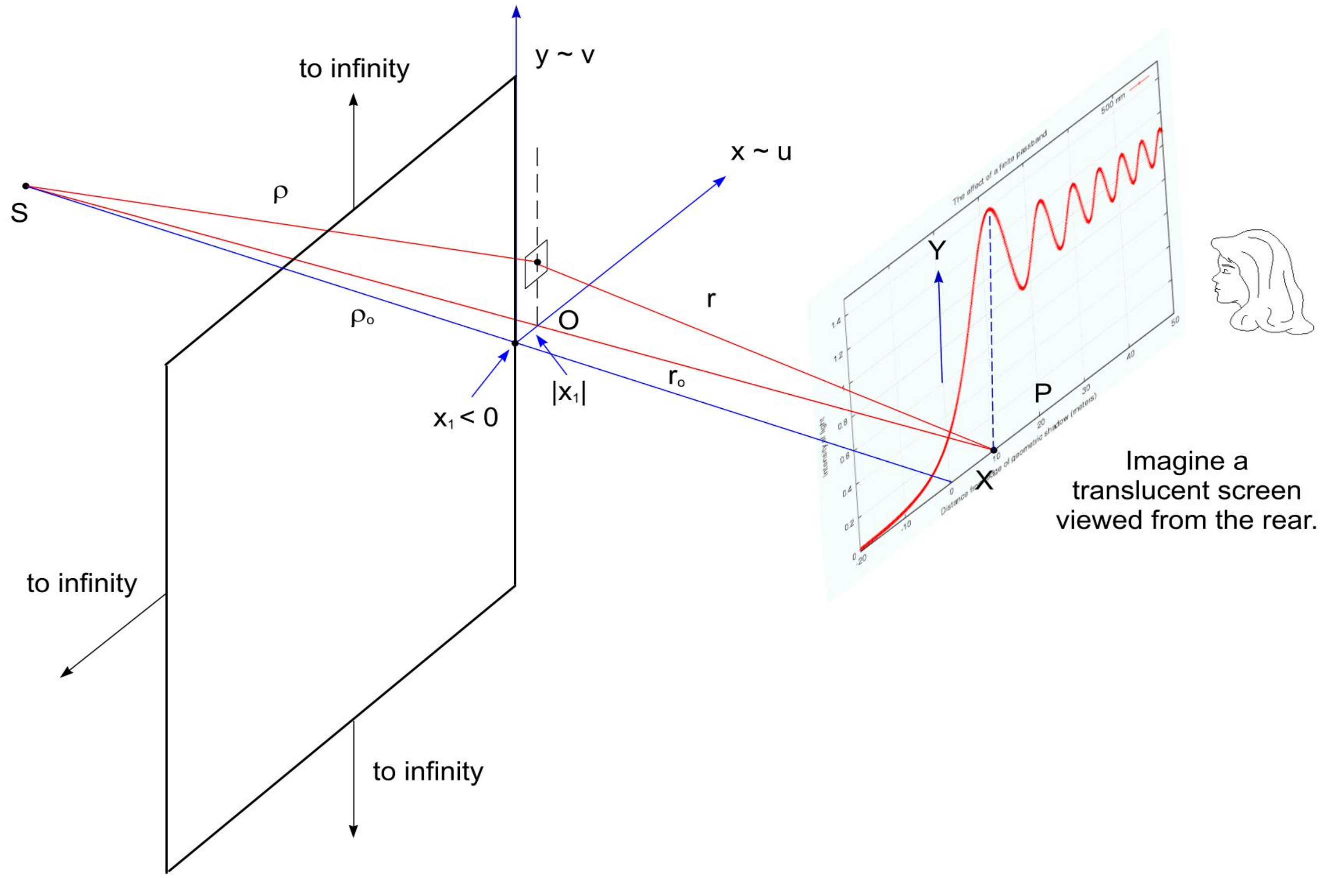
$$\left|E_p\right|=\frac{A_o}{\sqrt{2}}1.1\sqrt{2}=1.1A_o$$

For the irradiance comparison, there is not need to introduce that 1/2. Just square and compare!

$\left|E_p\right|^2=1.1^2A_o^2=1.2A_o^2$. There is the 1.2.

If we keep going max, min, max, min, etc., u keeps getting more and more negative approaching negative infinity. The differences between maxes and mins decrease and we eventually approach 1, the unobstructed wave. We are far enough away from the shadow now.

Fringe Spacing for the Lunar Occultation: What is X?

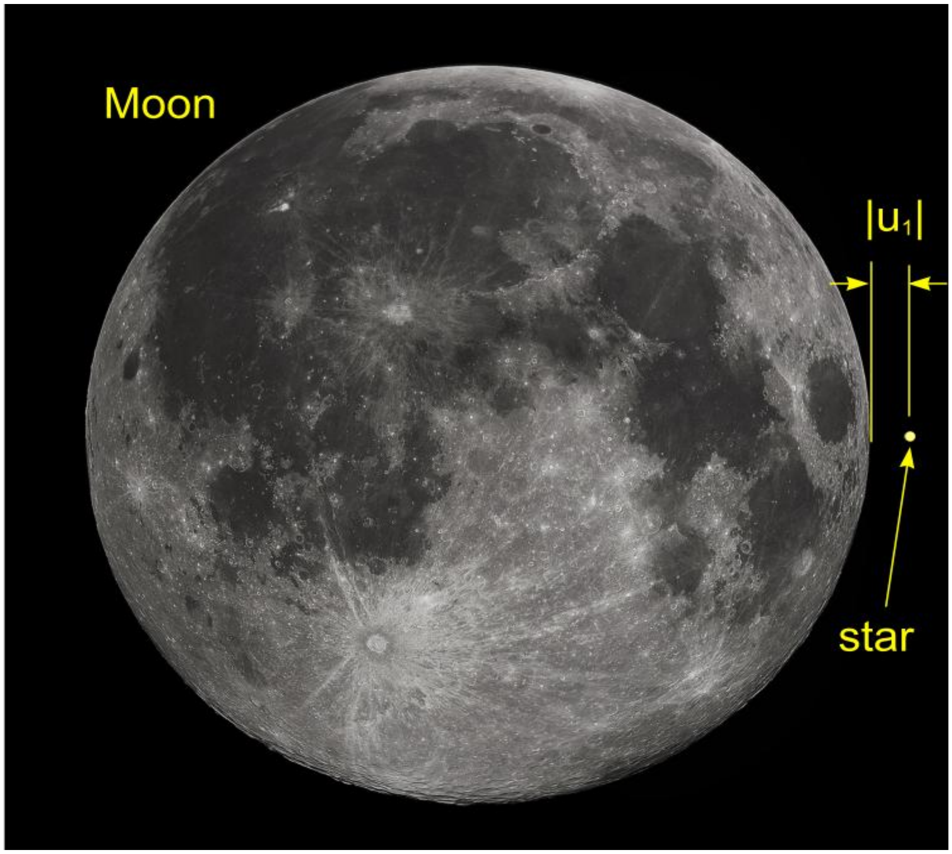


From similar triangles: $\frac{|x_1|}{\rho_o} = \frac{X}{\rho_o + r_o}$ => $X = \frac{\rho_o + r_o}{\rho_o} |x_1|$.

For a distant star, $\lim_{\rho_o \to \infty} X = \lim_{\rho_o \to \infty} (1 + r_o / \rho_o) |x_1| = |x_1|$.

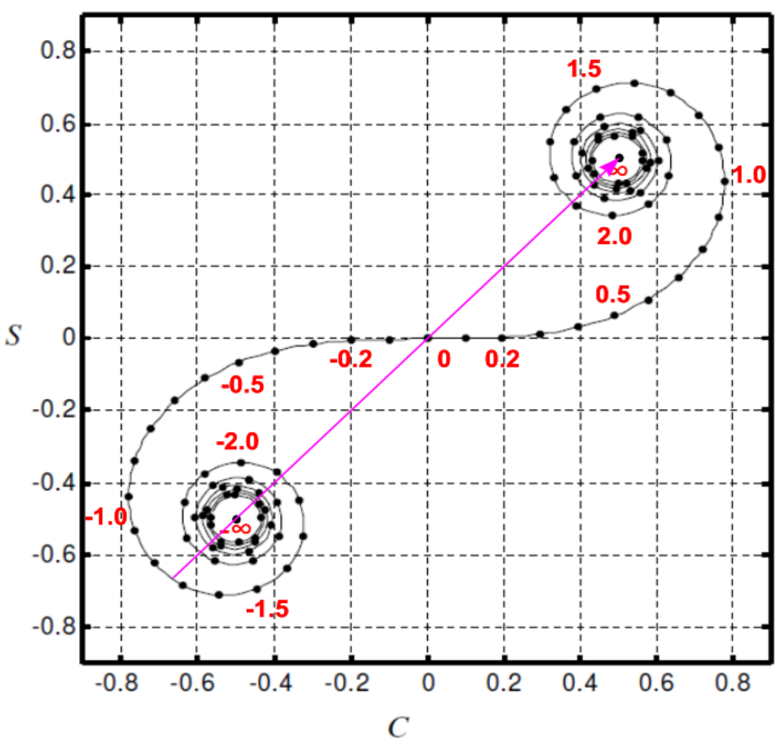
Remember the integration is $E_p = \frac{A_o}{\sqrt{2}} \int_{u_1}^{\infty} e^{i\pi u^2/2} du$, where $u = x \left[\frac{2(\rho_o + r_o)}{\lambda \rho_o r_o} \right]^{1/2}$.

$$\lim_{\rho_o \to \infty} u = x \lim_{\rho_o \to \infty} \left[\frac{2(\rho_o + r_o)}{\lambda \rho_o r_o} \right]^{1/2} = x \left[\frac{2}{\lambda r_o} \right]^{1/2}$$



Moon from Wikipedia: Gregory H. Revera. Creative Commons

So now we need to know that u that gave us the maximum.



We now need the u-value. From the Cornu spiral

$$u = 1.3 .$$

Then we need

$$u = x\left[\frac{2}{\lambda r_o}\right]^{1/2}$$

and finally $X = x$, giving

$$X = u\left[\frac{\lambda r_o}{2}\right]^{1/2}$$

The values are: $\lambda = 500 \text{ nm}$, $u = 1.3$, $r_o = 380,000 \text{ km}$,
where the last parameter is the Moon-Earth distance,

$$r_o = 380,000 \text{ km} = 3.8 \times 10^8 \text{ m}.$$

$$X = |u| \left[\frac{\lambda r_o}{2} \right]^{1/2} = 1.3 \left[\frac{500 \times 10^{-9} \text{ m} \cdot 3.8 \times 10^8 \text{ m}}{2} \right]^{1/2}$$

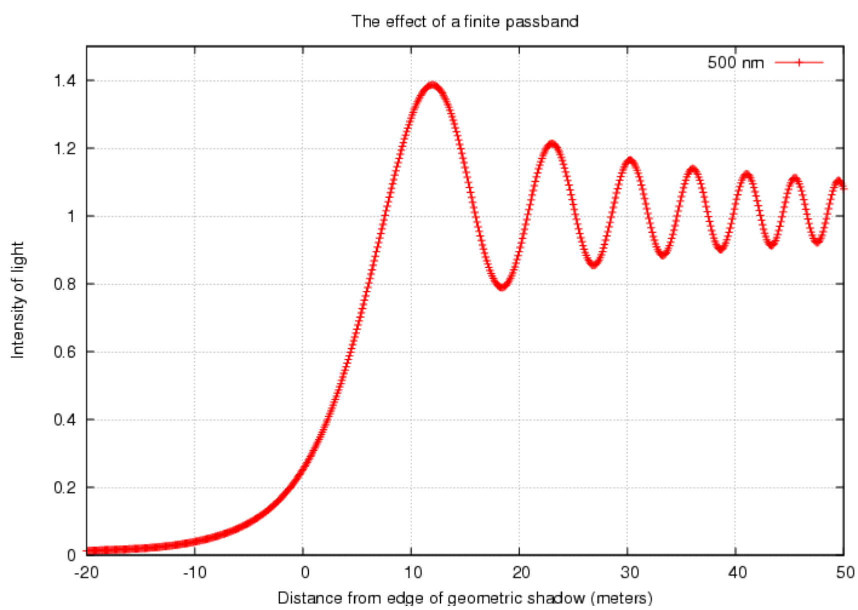
$$X = 1.3 \left[\frac{500 \times 10^{-1} \cdot 3.8}{2} \right]^{1/2}$$

$$X = 1.3 \left[\frac{50 \cdot 3.8}{2} \right]^{1/2} = 1.3 [25 \cdot 3.8]^{1/2} = 12.7 \text{ m}$$

or without a calculator,

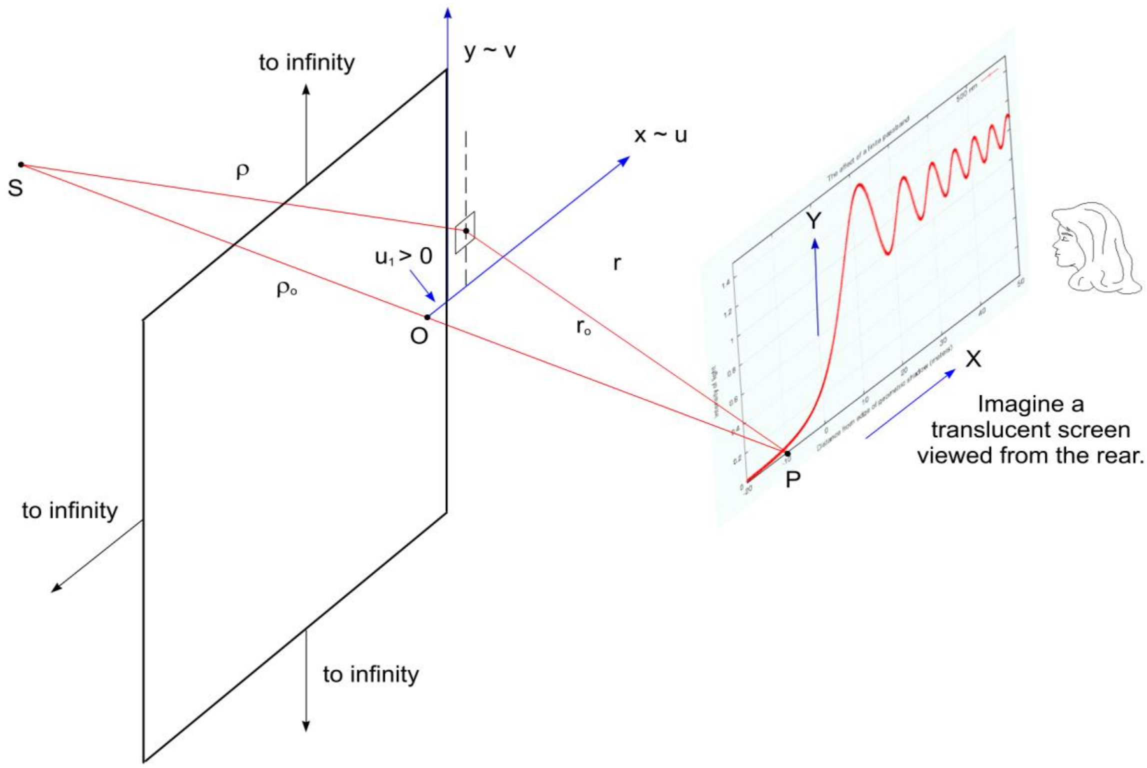
$$X \approx 1.3 [25 \cdot 4]^{1/2} = 1.3 [100]^{1/2} = 1.3 \cdot 10 = 13 \text{ m}.$$

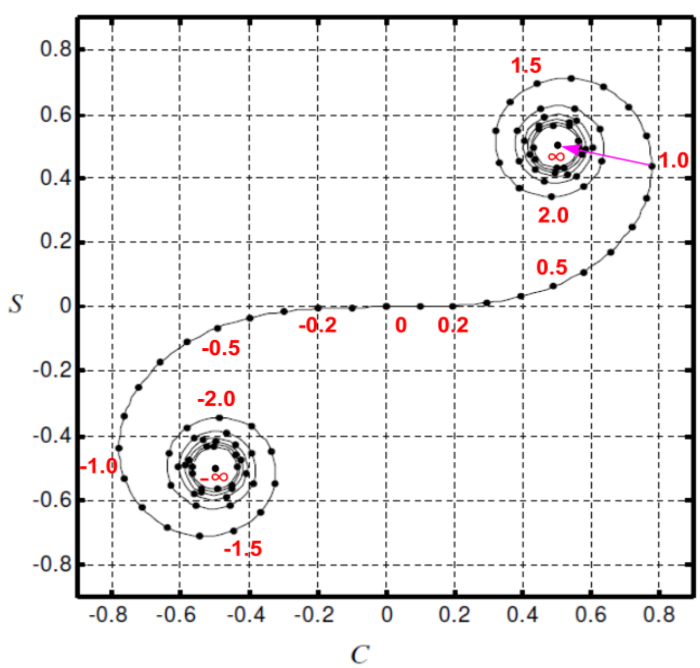
Check out the peak below, which is a little to the right of 10 m.



W3. What about in the shadow region: the exponential looking tail?

See the above figure that drops from 0.25 at X = 0 to zero to the left of X = 0. Now $u_1 > 0$





Here is one case in the shadow region.

$$E_p = \frac{A_o}{\sqrt{2}} \int_1^{\infty} e^{i\pi u^2/2} du$$

The integral part from the table is

$$[C(\infty) - C(1)] + i[S(\infty) - S(1)]$$

$$\approx [0.50 - 0.78] + i[0.50 - 0.44]$$

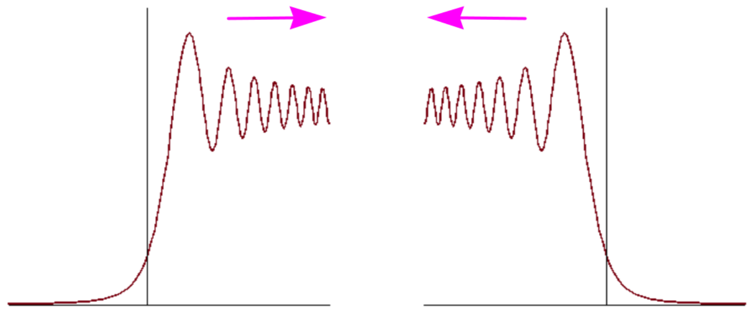
$$\left|E_p\right| = \frac{A_o}{\sqrt{2}} \sqrt{(-0.28)^2 + 0.06^2}$$

$$\left|E_p\right| = 0.20 A_o$$

$$\left|E_p\right|^2 = 0.04 A_o^2 \approx 0$$

Can you imagine increasing u from 0 to ∞ and watching the phasor shrink to zero!

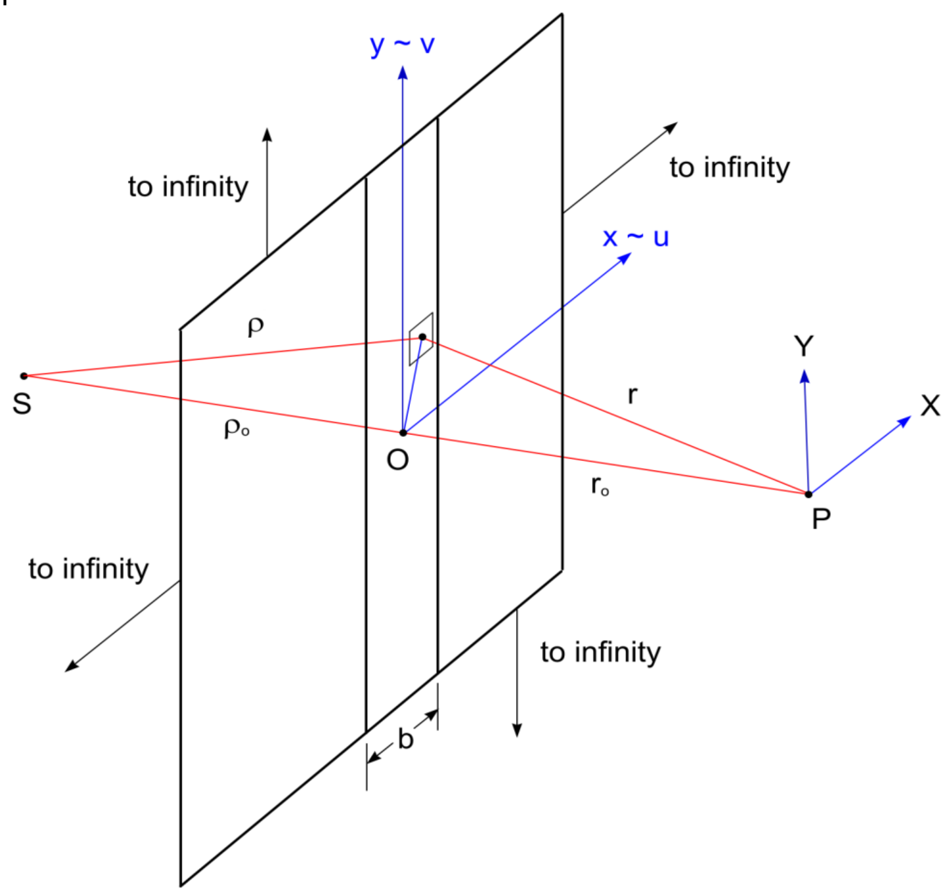
W4. Fresnel Single Slit. The trick is to bring two edges together from either side. Figure Courtesy Gisling, Wikipedia. Creative Commons.

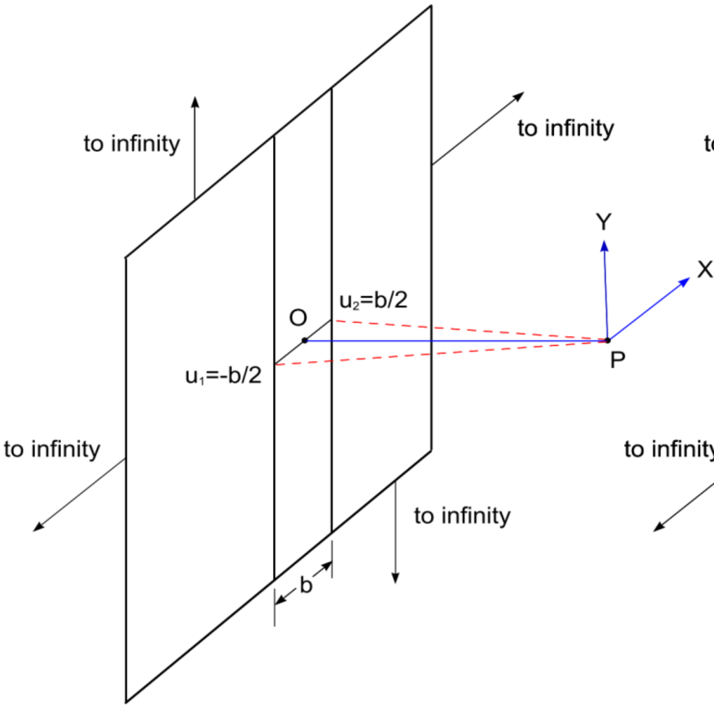


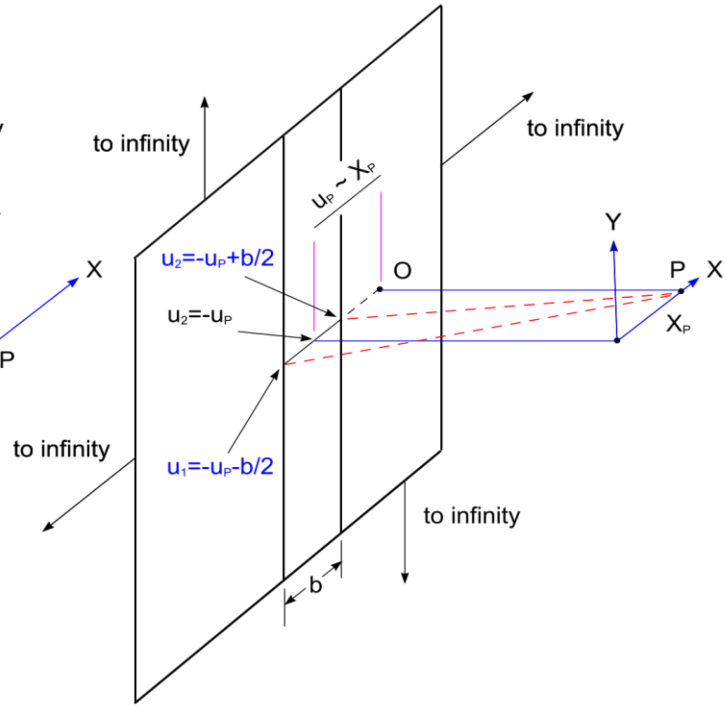
As you bring these two edges together, a variety of diffraction patterns can result depending on the slit width. The basic integral is

$$E_p = \frac{A_o}{2}\int_{u_1}^{u_2} e^{i\pi u^2/2}du \int_{-\infty}^{\infty} e^{i\pi v^2/2}dv \quad \Rightarrow \quad E_p = \frac{A_o}{\sqrt{2}}\int_{u_1}^{u_2} e^{i\pi u^2/2}du ,$$

where u_1 and u_2 depend on the point P we are investigating with one important constraint. This constraint is that $u_2 - u_1 = b$, the slit width. For the P along the main axis $u_1 = -b/2$ and $u_2 = +b/2$. For other points, the integration limits will not be symmetrically placed.



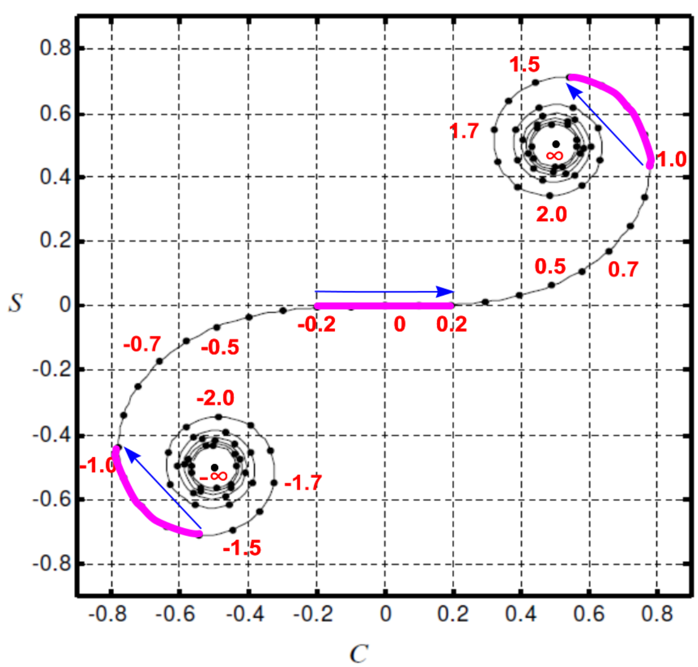




Left Figure: $$E_p = \frac{A_o}{\sqrt{2}} \int_{-b/2}^{+b/2} e^{i\pi u^2/2} du$$

Right Figure: $$E_p = \frac{A_o}{\sqrt{2}} \int_{-u_p-b/2}^{-u_p+b/2} e^{i\pi u^2/2} du$$

Note: Since the integration span is always the same, we come to Doc's SNAKE RULE.
Snake your way around the Cornu Spiral!

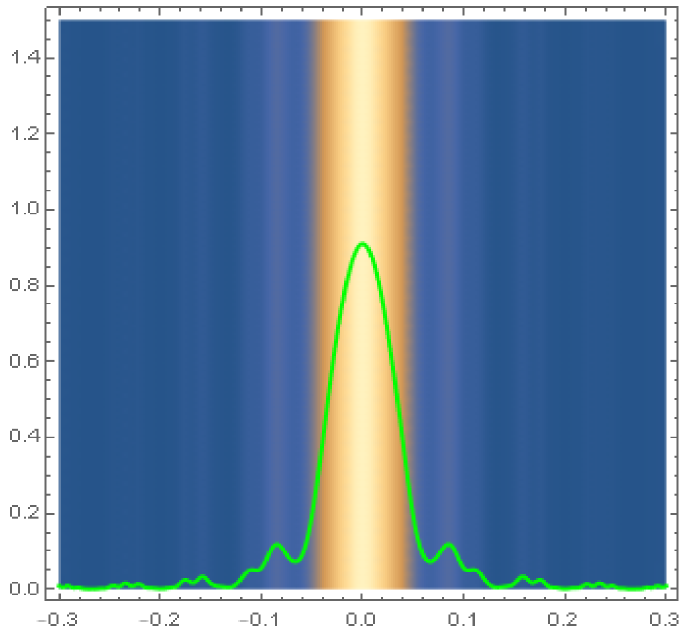


Baby Snake Found in
Doc's Basement (9/24/2011)

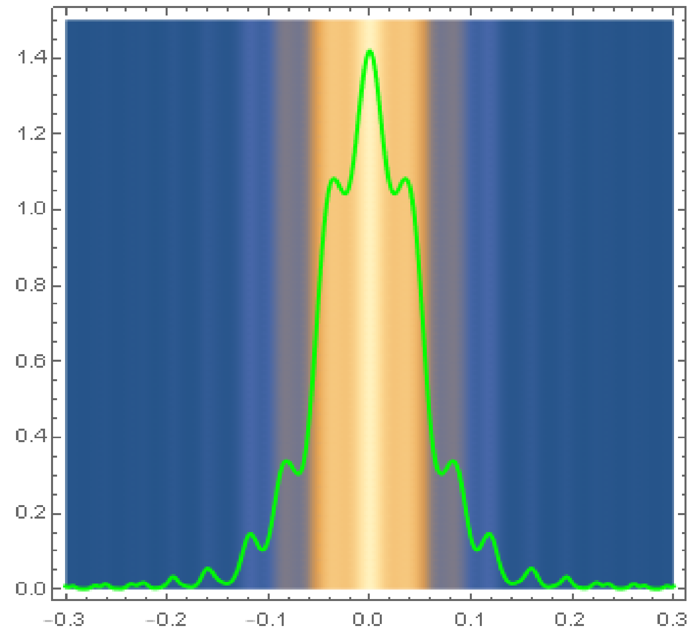
[Great Old PSI Houston Film Loop]
Video of the "Snake" in Action

Mathematica takes care of us, snakes around the Cornu spiral,
and computes several Fresnel slit diffraction patterns for us.

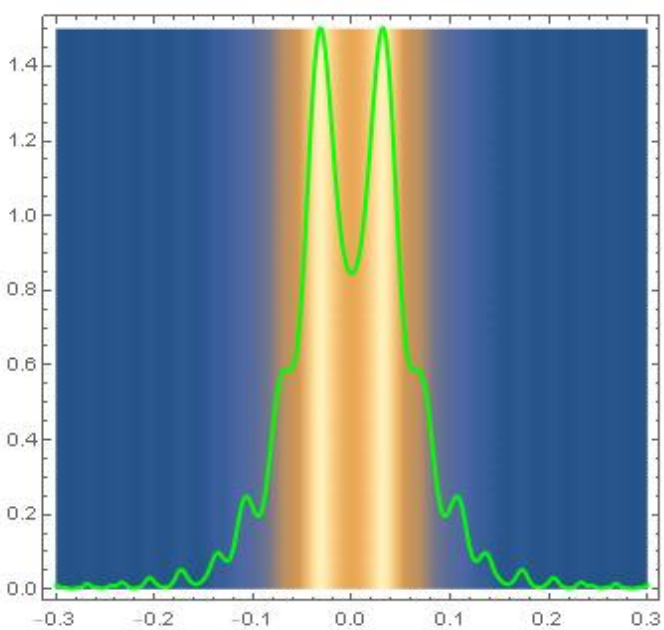
Relative Intensity versus Distance from Midpoint in mm, Using Mathematica.
Wavelength is 600 nm. Distance from Slit to Screen is 1 m.



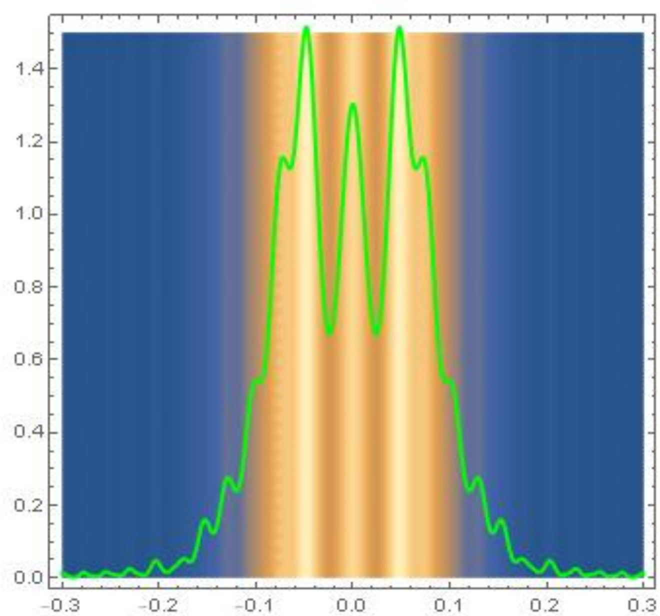
Slit Width = 0.09 mm



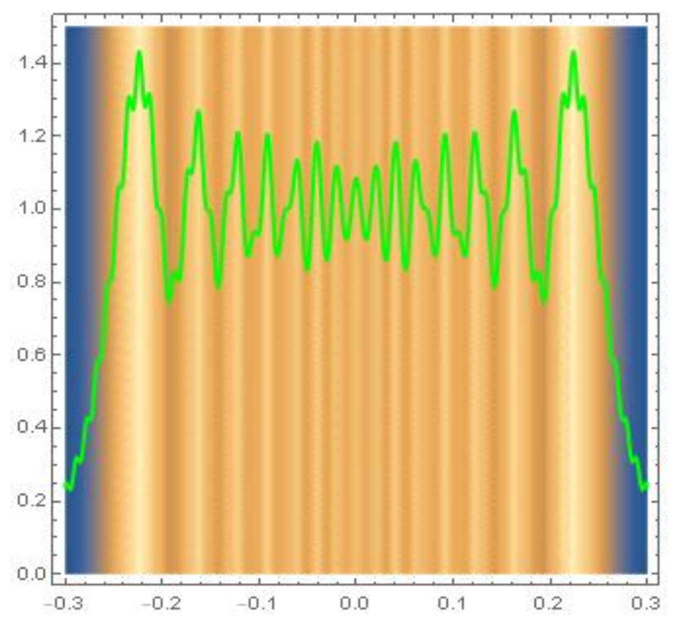
Slit Width = 0.17 mm



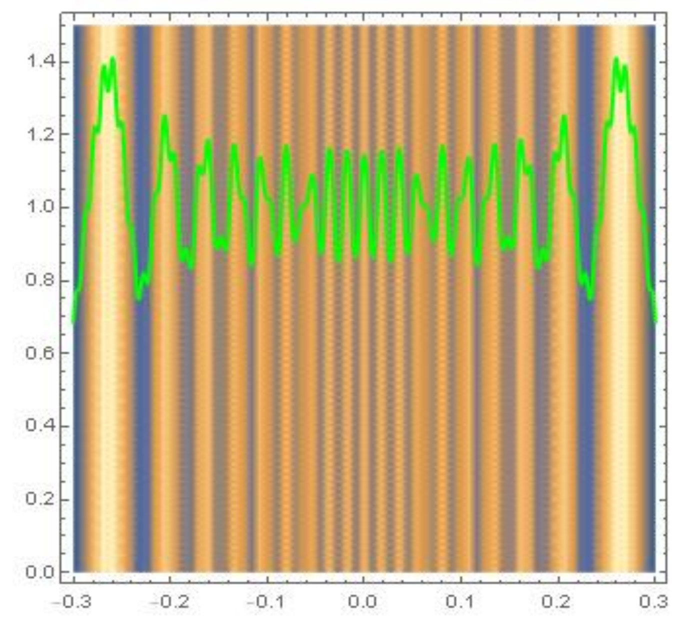
Slit Width = 0.19 mm



Slit Width = 0.25 mm



Slit Width = 0.59 mm



Slit Width = 0.67 mm

Can you imagine here with the large width the two edge patterns coming in from either side?

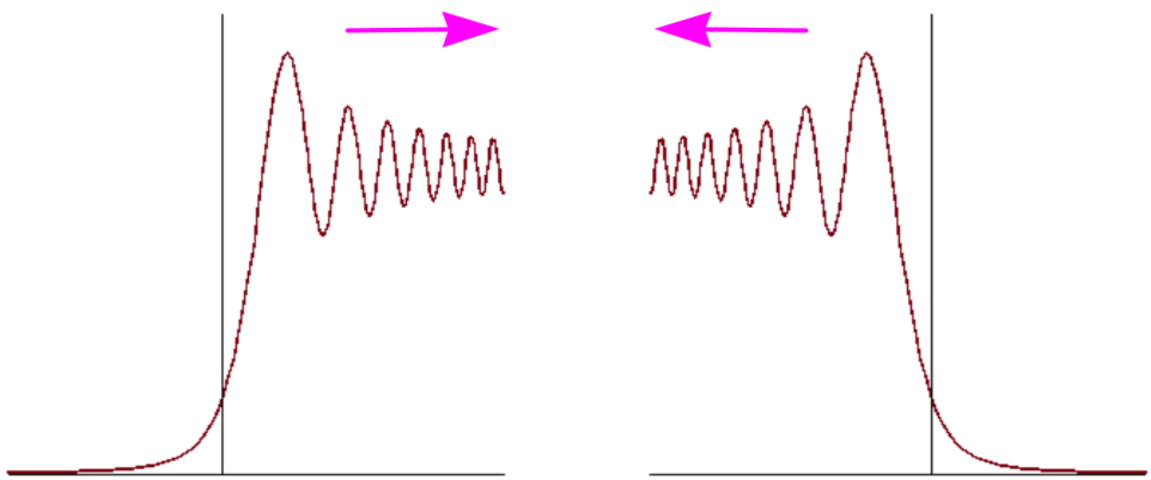


Figure Adapted from Wikipedia: Courtesy Gisling. Creative Commons.