Theoretical Physics
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Chapter 0 Homework. Intro Physics - SOLUTONS

HW-01. Combinatorics. How many ways can two of a family of five be chosen to go to the movies?

Method 1. Brute Force. Name the family members Alice, Bob, Carol, Diane, and Evan. Here are the choices.

Consider these 4: Alice – Bob, Alice – Carol, Alice – Diane, Alice – Evan
Then consider these 3: Bob – Carol, Bob -Diane, Bob – Evan
Then consider these 2: Carol – Diane, Carol - Evan
Then we have left 1: Diane – Evan

Total = 10

Method 2. Number of Ways Method.

There are 5 ways to pick the first person and then 4 ways to be the second. But since the order doesn't matter, you need to divide by 2. The answer is (5)(4)/2 = 10.

Method 3. Factorials. We want

$$\frac{5\cdot 4}{2} = \frac{5\cdot 4}{2\cdot 1} = \frac{5!}{2!3!}$$

For n people, where k are chosen to go to the movies, you have

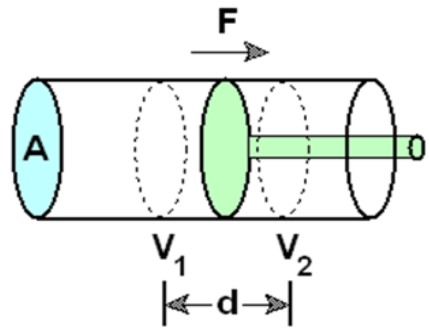
$$\frac{5\cdot 4}{2} = \frac{5\cdot 4}{2\cdot 1} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Method 4. Statistical Mechanics Logic (To Be Covered Later in the Course). There are n particles and two energy states. One energy state is going to the movies: say n_1 go to the movies. The energy state of staying home is state 2 with n_2. Then, the number of ways are

$$\frac{n!}{n_1!n_2!}$$

For three states with n people: some going to movies (n_1) , some staying at home (n_2), and some going to Spain (n_3), you would have $\dfrac{n!}{n_1!n_2!n_3!}$ number of ways.

HW-02. Work and the Ideal Gas. An ideal gas expands.



a) Work $W = \int F dx$. But pressure is force per area.

$$P = \frac{F}{A}$$

Therefore, $W = \int F dx = \int P A dx = \int P dV$.

b) Expansion at constant temperature T_0 . Use the ideal gas law $PV = nRT$.

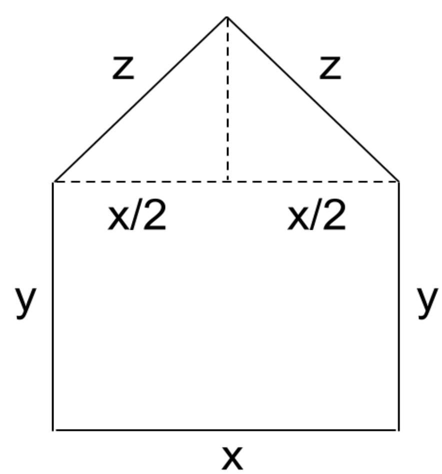
$$W = \int P dV => \int_{V_1}^{V_2} P dV = \int_{V_1}^{V_2} \frac{nRT_0}{V} dV$$

Pull out the constants.

$$W = nRT_0 \int_{V_1}^{V_2} \frac{1}{V} dV = nRT_0 \ln V \Big|_{V_1}^{V_2} = nRT_0 (\ln V_2 - \ln V_1)$$

$$W = nRT_0 \ln \frac{V_2}{V_1}$$

HW-03. Undetermined Multipliers I. You have 100 meters of fence and want to enclose the largest pentagon.



Soluton. The perimeter is $P = x + 2y + 2z$ and the area is $A = A(x, y, z) = A_{\text{rectangle}} + 2A_{\text{triamgle}}$.

$$A_{\text{rectangle}} = xy$$

For each small triangle: $A_{\text{triamgle}} = \frac{1}{2}\frac{x}{2}\sqrt{z^2 - (\frac{x}{2})^2}$,

i.e., 1/2 the base times the altitude.

$$A = A_{\text{rectangle}} + 2A_{\text{triamgle}} = xy + \frac{x}{2}\sqrt{z^2 - (\frac{x}{2})^2} = xy + \frac{x}{4}\sqrt{4z^2 - x^2}$$

$$A = xy + \frac{x}{4}\sqrt{4z^2 - x^2} \quad \text{and} \quad P = x + 2y + 2z$$

$$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial y}dz = 0 \qquad dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial y}dz = 0$$

$$dA - \lambda dP = (\frac{\partial A}{\partial x} - \lambda\frac{\partial P}{\partial x})dx + (\frac{\partial A}{\partial y} - \lambda\frac{\partial P}{\partial y})dy + (\frac{\partial A}{\partial z} - \lambda\frac{\partial P}{\partial z})dz = 0$$

The secret here is that λ can be chosen so that individually

$$\frac{\partial A}{\partial x} - \lambda\frac{\partial P}{\partial x} = 0 \qquad \frac{\partial A}{\partial y} - \lambda\frac{\partial P}{\partial y} = 0 \qquad \frac{\partial A}{\partial z} - \lambda\frac{\partial P}{\partial z} = 0 .$$

$$\frac{\partial A}{\partial x} = y + \frac{1}{4}\sqrt{4z^2 - x^2} + \frac{x}{4}\frac{1}{2}\frac{1}{\sqrt{4z^2 - x^2}}(-2x)$$

$$\frac{\partial A}{\partial x} = y + \frac{1}{4}\sqrt{4z^2 - x^2} - \frac{x^2}{4}\frac{1}{\sqrt{4z^2 - x^2}}$$

$$\frac{\partial A}{\partial x} = y + \frac{1}{4}\frac{1}{\sqrt{4z^2 - x^2}}\left[(4z^2 - x^2) - x^2\right]$$

$$\frac{\partial A}{\partial x} = y + \frac{1}{4}\frac{1}{\sqrt{4z^2 - x^2}}(4z^2 - 2x^2)$$

$$\boxed{\frac{\partial A}{\partial x} = y + \frac{2z^2 - x^2}{2\sqrt{4z^2 - x^2}}}$$

$$A = xy + \frac{x}{4}\sqrt{4z^2 - x^2} \quad \Rightarrow \quad \boxed{\frac{\partial A}{\partial y} = x}$$

$$\frac{\partial A}{\partial z} = \frac{x}{4}\frac{1}{2}\frac{1}{\sqrt{4z^2 - x^2}}4 \cdot 2z = \frac{xz}{\sqrt{4z^2 - x^2}} \quad \Rightarrow \quad \boxed{\frac{\partial A}{\partial z} = \frac{xz}{\sqrt{4z^2 - x^2}}}$$

$$P = x + 2y + 2z \quad \Rightarrow \quad \frac{\partial P}{\partial x} = 1 \qquad \frac{\partial P}{\partial y} = 2 \qquad \frac{\partial P}{\partial z} = 2$$

Summary for A: $$\frac{\partial A}{\partial x} = y + \frac{2z^2 - x^2}{2\sqrt{4z^2 - x^2}} \qquad \frac{\partial A}{\partial y} = x \qquad \frac{\partial A}{\partial z} = \frac{xz}{\sqrt{4z^2 - x^2}}$$

Answers.

$$\frac{\partial A}{\partial x} - \lambda \frac{\partial P}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial A}{\partial x} = \lambda \frac{\partial P}{\partial x} \quad \Rightarrow \quad \boxed{y + \frac{2z^2 - x^2}{2\sqrt{4z^2 - x^2}} = \lambda}$$

$$\frac{\partial A}{\partial y} - \lambda \frac{\partial P}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial A}{\partial y} = \lambda \frac{\partial P}{\partial y} \quad \Rightarrow \quad \boxed{x = 2\lambda}$$

$$\frac{\partial A}{\partial z} - \lambda \frac{\partial P}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial A}{\partial z} = \lambda \frac{\partial P}{\partial z} \quad \Rightarrow \quad \boxed{\frac{xz}{\sqrt{4z^2 - x^2}} = 2\lambda}$$

HW-04. Undetermined Multipliers II. Completing the solution. This particular problem is pure algebra. We want to solve for 4 unknowns (x, y, z, and λ) where we have 4 equations.

Starting Point: $$y + \frac{2z^2 - x^2}{2\sqrt{4z^2 - x^2}} = \lambda \qquad x = 2\lambda \qquad \frac{xz}{\sqrt{4z^2 - x^2}} = 2\lambda$$

Set the last two equations equal. $x = \frac{xz}{\sqrt{4z^2 - x^2}}$. Then $\frac{xz}{\sqrt{4z^2 - x^2}} - x = 0$.

We get two solutions from $x\left[\frac{z}{\sqrt{4z^2 - x^2}} - 1\right] = 0$.

We don’t care about x = 0 as that solution would lead to a minimum case. Therefore,

$$\frac{z}{\sqrt{4z^2 - x^2}} = 1$$

$$z = \sqrt{4z^2 - x^2} \quad => \quad z^2 = 4z^2 - x^2 \quad => \quad 3z^2 = x^2 \quad => \quad z = \frac{x}{\sqrt{3}} > 0 .$$

Then $x = 2\lambda$ leads to $z = \frac{2\lambda}{\sqrt{3}}$.

For the y equation $y + \frac{2z^2 - x^2}{2\sqrt{4z^2 - x^2}} = \lambda$. First insert $x^2 = 3z^2$.

$$y + \frac{2z^2 - 3z^2}{2\sqrt{4z^2 - 3z^2}} = \lambda \quad => \quad y - \frac{z^2}{2\sqrt{z^2}} = \lambda \quad => \quad y - \frac{z}{2} = \lambda \quad => \quad y = \lambda + \frac{z}{2}$$

Use $z = \frac{2\lambda}{\sqrt{3}}$ to get $y = \lambda + \frac{1}{2}\frac{2\lambda}{\sqrt{3}} = \lambda + \frac{\lambda}{\sqrt{3}} = (1 + \frac{1}{\sqrt{3}})\lambda = (\frac{3+\sqrt{3}}{3})\lambda$.

Summary: $x = 2\lambda$, $y = (\frac{3+\sqrt{3}}{3})\lambda$, and $z = \frac{2\lambda}{\sqrt{3}}$.

Four unknowns need four equations. Our fourth equation is $P = x + 2y + 2z = 100$.

Therefore $P = x + 2y + 2z$ => $P = 2\lambda + 2(\frac{3+\sqrt{3}}{3})\lambda + 2\frac{2\lambda}{\sqrt{3}}$

$$P = \left[2 + 2(\frac{3+\sqrt{3}}{3}) + 4\frac{1}{\sqrt{3}}\right]\lambda \quad => \quad P = \left[2 + 2 + \frac{2}{3}\sqrt{3} + 4\frac{1}{\sqrt{3}}\right]\lambda$$

$$P = \left[4 + \frac{2}{3}\sqrt{3} + 4\frac{\sqrt{3}}{3}\right]\lambda \quad => \quad P = \left[4 + \frac{6}{3}\sqrt{3}\right]\lambda \quad => \quad P = \left[4 + 2\sqrt{3}\right]\lambda$$

$$P = 2\left[2 + \sqrt{3}\right]\lambda \quad => \quad \lambda = \frac{P}{2(2+\sqrt{3})}$$

$$\lambda = \frac{P}{2(2+\sqrt{3})}\left[\frac{2-\sqrt{3}}{2-\sqrt{3}}\right] \quad => \quad \lambda = \frac{(2-\sqrt{3})P}{2(2^2-\sqrt{3}^2)}$$

$$\lambda = \frac{(2-\sqrt{3})P}{2(4-3)} \quad => \quad \boxed{\lambda = \frac{(2-\sqrt{3})}{2}P}$$

The values $x = 2\lambda$, $y = (\frac{3+\sqrt{3}}{3})\lambda$, and $z = \frac{2\lambda}{\sqrt{3}}$ then become

$$\boxed{x = (2-\sqrt{3})P}$$

$$y = (\frac{3+\sqrt{3}}{3})\frac{(2-\sqrt{3})}{2}P = \frac{6-\sqrt{3}-3}{6}P = (\frac{3-\sqrt{3}}{6})P \quad => \quad \boxed{y = (\frac{3-\sqrt{3}}{6})P}$$

$$z = \frac{2}{\sqrt{3}}\frac{(2-\sqrt{3})}{2}P = \frac{(2-\sqrt{3})}{\sqrt{3}}P = \frac{(2\sqrt{3}-3)}{3}P \quad => \quad \boxed{z = \frac{(2\sqrt{3}-3)}{3}P}$$

Since P = 100 meters,

$$x = (2-\sqrt{3})P = 100(2-\sqrt{3}) = 26.79 \text{ meters} \quad => \quad \boxed{x = 26.79 \text{ meters}}$$

$$y = \frac{(3-\sqrt{3})}{6}P = 21.13 \text{ meters} \quad => \quad \boxed{y = 21.13 \text{ meters}}$$

$$z = \frac{(2\sqrt{3}-3)}{3}P = 15.47 \text{ meters} \quad => \quad \boxed{z = 15.47 \text{ meters}}$$

Check:

$$P = x+2y+2z = 26.79+2(21.13)+2(15.47) = 99.99 \;=\; 100.0 \text{ meters}$$