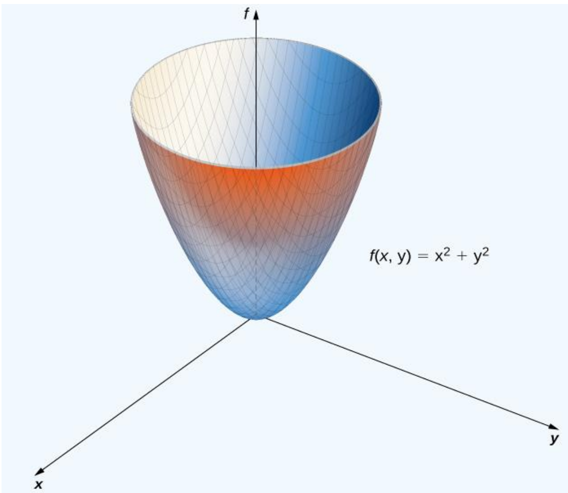
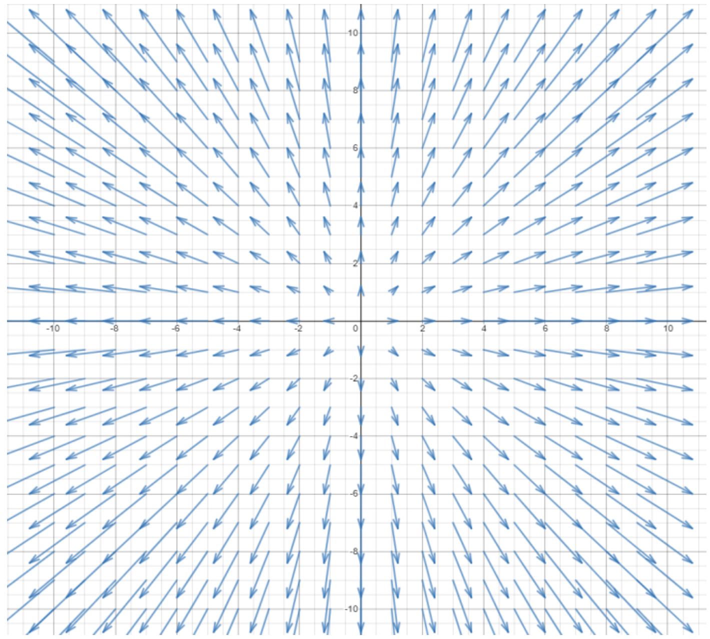
Theoretical Physics
Prof. Ruiz, UNC Asheville
Chapter E Homework. Differential Form for the Maxwell Equations

HW-E1. Gradient.





(a) Determine $\vec{A}(x,y) = \nabla g(x,y)$ where $g(x,y) = x^2 + y^2$.

$$\vec{A}(x,y) = (\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k})(x^2 + y^2) = 2x\,\hat{i} + 2y\,\hat{j}$$

$$\boxed{\vec{A}(x,y) = 2x\,\hat{i} + 2y\,\hat{j}}$$

(b) Calculate $\vec{B}(x,y) = \nabla f(x,y)$ where $f(x,y) = x^2 y$.

$$\vec{B}(x,y) = (\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k})(x^2 y) = 2xy\,\hat{i} + x^2\,\hat{j}$$

$$\boxed{\vec{B}(x,y) = 2xy\,\hat{i} + x^2\,\hat{j}}$$

(c) Find $\vec{B}(3,2) = \nabla f(3,2)$. $\vec{B}(3,2) = 2\cdot 3\cdot 2\,\hat{i} + 3^2\,\hat{j}$ $\boxed{\vec{B}(x,y) = 12\,\hat{i} + 9\,\hat{j}}$

(d) $D_u f(3,2) = ?$ for a direction parallel to the vector $\vec{V} = \hat{i} + 2\hat{j}$.

We use $D_u f(x,y) \equiv \nabla f(x,y) \cdot \hat{u}$. First find the unit vector parallel to $\vec{V} = \hat{i} + 2\hat{j}$. Since the length of $\vec{V}$ is $\sqrt{1^2 + 2^2} = \sqrt{5}$, the unit vector parallel to $\vec{V}$ is $\hat{u} = \frac{1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{j}$.

$$D_u f(3,2) \equiv \nabla f(3,2) \cdot \hat{u} = \vec{B}(3,2) \cdot \hat{u}$$

$$D_u f(3,2) = (12\hat{i} + 9\hat{j}) \cdot \hat{u}$$

$$D_u f(3,2) = (12\hat{i} + 9\hat{j}) \cdot (\frac{1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{j})$$

$$D_u f(3,2) = \frac{12}{\sqrt{5}} + \frac{9 \cdot 2}{\sqrt{5}}$$

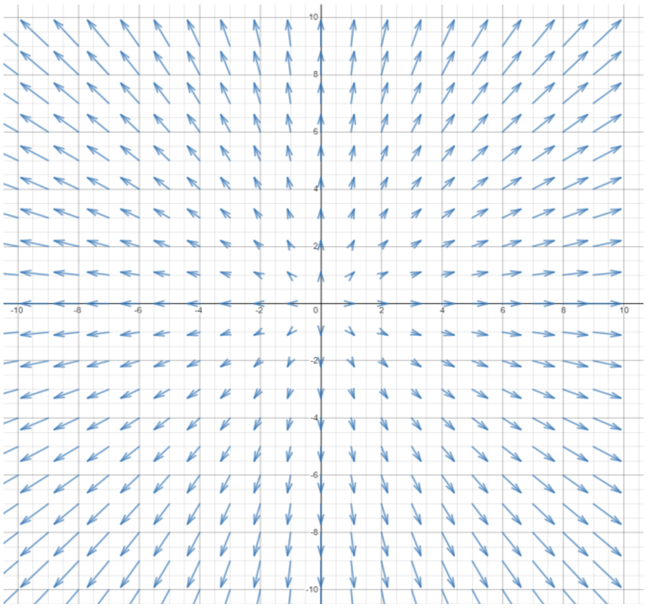
$$D_u f(3,2) = \frac{30}{\sqrt{5}}$$

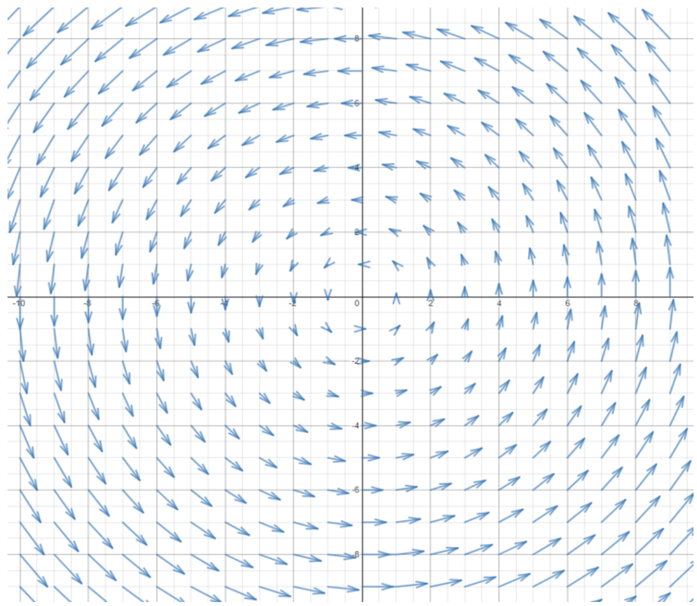
$$D_u f(3,2) = \frac{30}{5}\sqrt{5}$$

$$D_u f(3,2) = 6\sqrt{5}$$

Credits: Mathematics at LibreTexts for the paraboloid, desmos.com graphing for the vector figure, and Duane Q. Nykamp for gradient applications at mathinsight.org.

HW-E2. Divergence and Curl. The vector field $\vec{A}(x, y) = x\,\hat{i} + y\,\hat{j}$ appears in the left figure and $\vec{B}(x, y) = -y\,\hat{i} + x\,\hat{j}$ appears in the right figure below.





Courtesy desmos.com graphing.

Calculate the divergence and curl for each of these vector fields.

Give a description in words for the divergence and curl based on your results and the visualizations of the two fields.

$$\nabla \cdot \vec{A}(x, y) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(x\,\hat{i} + y\,\hat{j}\right) = \frac{\partial x}{\partial x}\hat{i} \cdot \hat{i} + \frac{\partial y}{\partial y}\hat{j} \cdot \hat{j}$$