Theoretical Physics
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Chapter I Homework. Maxwell-Boltzmann Velocity Distribution (Classical)

HW-I1. Maxwell-Boltzmann Velocity Distribution. We are going to do some cool and important classical stuff here. We know the following integral from earlier in our course.

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}\text{ . Therefore, } \int_0^{\infty} e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\,.$$

Also note that

$$\int_0^{\infty} x^2 e^{-\alpha x^2}\,dx = -\frac{d}{d\alpha}\left[\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\right] = \frac{1}{4\alpha}\sqrt{\frac{\pi}{\alpha}}\,.$$

You can simply just use the above integrals when needed below since we have derived the results in our course and above. The velocity probability distribution for particles in a gas moving only in one dimension can be written as

$$f(v_x) = Ae^{-\beta E_x}\text{ , where } \beta = \frac{1}{kT} \text{ from class and } E_x = \frac{1}{2}mv_x^2\,.$$

(a) Show that the normalization constant $A = \sqrt{\frac{m}{2\pi kT}}$ from $\int_{-\infty}^{+\infty} f(v_x)dv_x = 1$.

(b) In 3D, $f(v_x)dv_x\, f(v_y)dv_y\, f(v_z)dv_z = Ae^{-\beta E_x} Ae^{-\beta E_y} Ae^{-\beta E_z} dv_x dv_y dv_z$.

Confirm by integration in spherical velocity coordinates (v,θ,ϕ) that it integrates to 1.

HINT: $dv_x dv_y dv_z$ gets replaced by $v^2 \sin\theta\, dv\, d\theta\, d\phi$ where the v integration goes from 0 to ∞, the θ integration goes from 0 to π, and the ϕ integration goes from 0 to 2π .

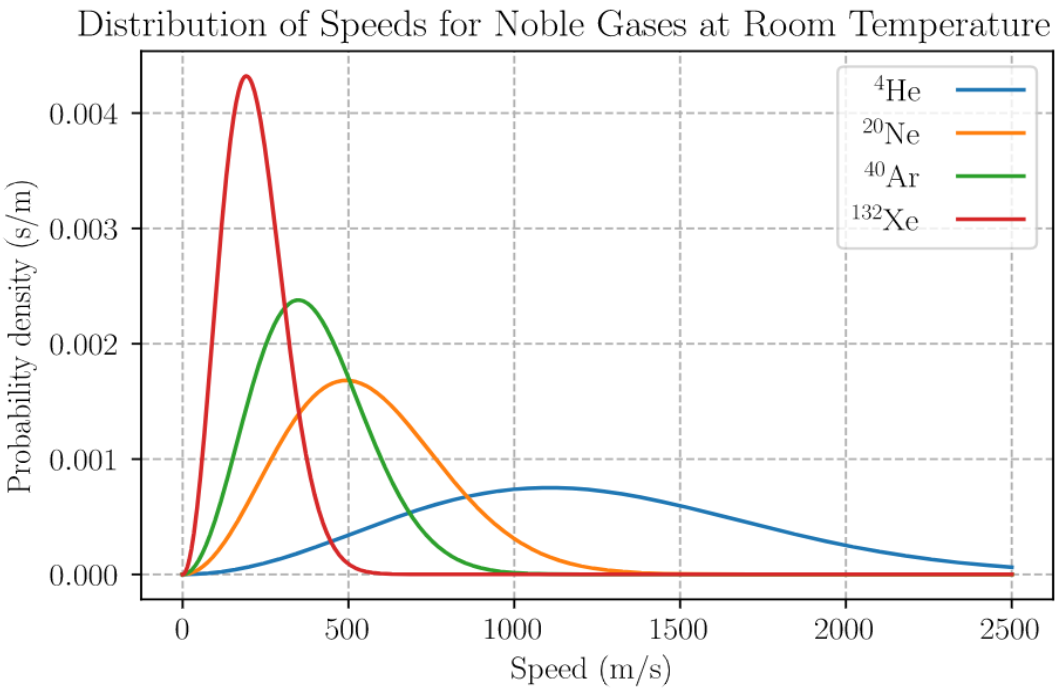
(c) Show that for $\int_0^{\infty} f(v)dv = 1$ you must have $f(v) = 4\pi\left[\frac{m}{2\pi kT}\right]^{3/2} v^2 e^{-\frac{mv^2}{2kT}}$.

Note you have found what this function specifically is, the function $f(v)$ we encountered in the abstract in "deriving" the ideal gas law. **HINT:** Look at your work for Part (b). What is left after you integrate over the angles but have not yet integrated over v? All that stuff in the integrand is f(v). But be sure to express the constant A^3 in terms of k, T, π, etc.

HW-I2. The Most Probable Speed. Graphs of the Maxwell-Boltzmann velocity distribution

$$f(v) = 4\pi \left[\frac{m}{2\pi kT} \right]^{3/2} v^2 e^{-\frac{mv^2}{2kT}}$$

are show below. There are four specific graphs.



In general, show that the most probable speed is given by the formula

$$v_p = \sqrt{\frac{2kT}{m}} .$$

HINT: The most probably speed occurs where the probability function is maximum. You know how to do such a problem from calculus – the so-called max-min problem.

HW-I3. The Average Speed. In general, show that the average speed for a particle with the Maxwell-Boltzman velocity distribution is given by

$$\bar{v} = \sqrt{\frac{8kT}{\pi m}}$$

HINT: The average is given by $\bar{v} = \int_0^\infty v f(v) dv$.

HW-I4. The Root-Mean-Square Speed. The root-mean-square speed is defined as the square root of the average of the velocity squared. In general, show that the root-mean-square speed for a particle with the Maxwell-Boltzman velocity distribution is

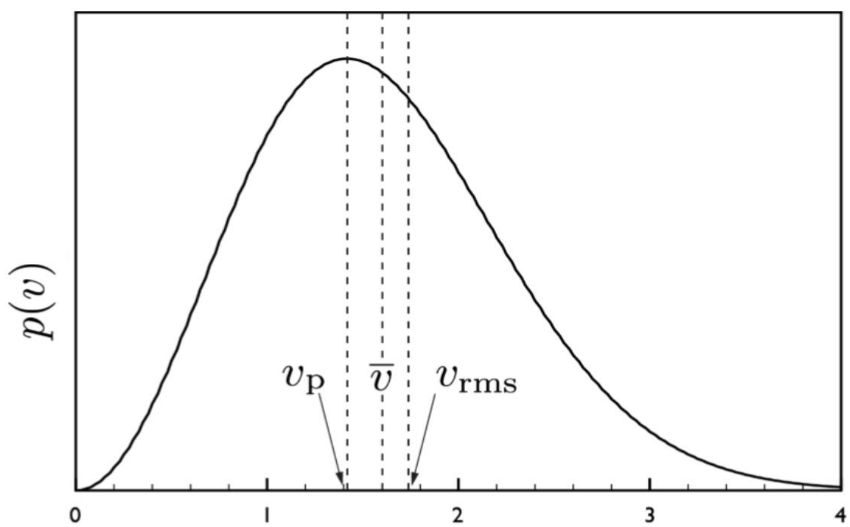
$$v_{rms} = \sqrt{\frac{3kT}{m}}$$

HINT: The average of the velocity square is given by

$$\overline{v^2} = \int_0^{\infty} v^2 f(v) dv$$

and $v_{rms} = \sqrt{\overline{v^2}}$.

HW-I5. The Big Three. Show that $v_p < \bar{v} < v_{rms}$. What do the numbers along the horizontal axis in the physics.stackexchange.com graph below represent?



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