Theoretical Physics
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Chapter I Homework. Quantum Mechanics

HW-I1. Maxwell-Boltzmann Velocity Distribution (Classical). We know the following integral from earlier in our course.

Useful integrals: $\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}}$, $\int_0^{\infty} e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$.

$$\int_0^{\infty} x^2 e^{-\alpha x^2}\, dx = -\frac{d}{d\alpha}\left[\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\right] = \frac{1}{4\alpha}\sqrt{\frac{\pi}{\alpha}} .$$

The velocity probability distribution for particles in a gas moving only in one dimension can be written as

$f(v_x) = A_x e^{-\beta E_x}$, where $\beta = \frac{1}{kT}$ from class and $E_x = \frac{1}{2} m v_x^2$.

(a) Find the normalization constant A from $\int_{-\infty}^{+\infty} f(v_x) dv_x = 1$.

$$1 = \int_{-\infty}^{+\infty} f(v_x) dv_x = \int_{-\infty}^{+\infty} A e^{-\beta E_x} dv_x = A \int_{-\infty}^{+\infty} e^{-\frac{m}{2kT} v_x^2} dv_x$$

Now use $\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}}$ where $\alpha = \frac{m}{2kT}$.

$$A \int_{-\infty}^{+\infty} e^{-\frac{m}{2kT} v_x^2} dv_x = A\sqrt{\frac{\pi}{m/(2KT)}} = A\sqrt{\frac{2\pi kT}{m}} = 1$$

$$A = \sqrt{\frac{m}{2\pi kT}}$$

(b) In 3D, $f(v_x)dv_x f(v_y)dv_y f(v_z)dv_z = A e^{-\beta E_x} A e^{-\beta E_y} A e^{-\beta E_z} dv_x dv_y dv_z$.

Confirm by integration in spherical velocity coordinates (v,θ,ϕ) that you get 1.

$$f(v_x)dv_x\, f(v_y)dv_y\, f(v_z)dv_z \rightarrow A^3 e^{-\frac{m}{2kT}(v_x^2+v_y^2+v_z^2)} v^2 \sin\theta dv d\theta d\phi$$

$$= \left[\frac{m}{2\pi kT}\right]^{3/2} e^{-\frac{m}{2kT}v^2} v^2 \sin\theta dv d\theta d\phi$$

$$I \equiv \left[\frac{m}{2\pi kT}\right]^{3/2} \int_{v=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} e^{-\frac{m}{2kT}v^2} v^2 \sin\theta dv d\theta d\phi$$

$$I = \left[\frac{m}{2\pi kT}\right]^{3/2} \int_{v=0}^{\infty} e^{-\frac{m}{2kT}v^2} v^2 dv \int_{\theta=0}^{\pi} \sin\theta d\theta \int_{\phi=0}^{2\pi} d\phi$$

First do this integral: $\int_{\phi=0}^{2\pi} d\phi = \phi\Big|_0^{2\pi} = 2\pi - 0 = 2\pi$.

$$I = 2\pi \left[\frac{m}{2\pi kT}\right]^{3/2} \int_{v=0}^{\infty} e^{-\frac{m}{2kT}v^2} v^2 dv \int_{\theta=0}^{\pi} \sin\theta d\theta$$

Next do $\int_{\theta=0}^{\pi} \sin\theta d\theta = -\cos\theta\Big|_0^{\pi} = -\cos\pi - (-\cos 0) = 1 - (-1) = 2$.

$$I = 4\pi \left[\frac{m}{2\pi kT}\right]^{3/2} \int_{v=0}^{\infty} e^{-\frac{m}{2kT}v^2} v^2 dv$$

Finally use $\int_0^{\infty} x^2 e^{-\alpha x^2} dx = -\frac{d}{d\alpha}\left[\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\right] = \frac{1}{4\alpha}\sqrt{\frac{\pi}{\alpha}}$ where $\alpha = \frac{m}{2kT}$.

$$I = 4\pi \left[\frac{m}{2\pi kT}\right]^{3/2} \frac{1}{4\alpha}\sqrt{\frac{\pi}{\alpha}} \text{ with } \alpha = \frac{m}{2kT} .$$

$$I = 4\pi\left[\frac{m}{2\pi kT}\right]^{3/2}\frac{1}{2}\frac{kT}{m}\sqrt{\frac{2\pi kT}{m}}$$

$$I = \left[\frac{m}{2\pi kT}\right]^{3/2}\frac{2\pi kT}{m}\sqrt{\frac{2\pi kT}{m}}$$

$$I = \left[\frac{m}{2\pi kT}\right]^{3/2}\left[\frac{2\pi kT}{m}\right]^{3/2} = 1$$

The result is 1 as expected.

(c) What is $f(v)$ in terms of v and the simplest constants such that $\int_0^\infty f(v)dv = 1$

Pick up with $I = 4\pi\left[\frac{m}{2\pi kT}\right]^{3/2}\int_{v=0}^{\infty} e^{-\frac{m}{2kT}v^2} v^2 dv = 1$ from the last section.

Compare with $\int_0^\infty f(v)dv = 1$.

$$f(v) = 4\pi\left[\frac{m}{2\pi kT}\right]^{3/2} e^{-\frac{m}{2kT}v^2} v^2$$

$$f(v) = 4\pi\left[\frac{m}{2\pi kT}\right]^{3/2} v^2 e^{-\frac{m}{2kT}v^2}$$

HW-I2. The Most Probable Speed.

We want to find the speed where the function f(v) has its maximum.

$$f(v) = 4\pi\left[\frac{m}{2\pi kT}\right]^{3/2} v^2 e^{-\frac{m}{2kT}v^2}$$

$$\frac{df(v)}{dv} = 0$$

$$\frac{d}{dv}\left[v^2 e^{-\frac{m}{2kT}v^2}\right] = 0$$

$$2ve^{-\frac{m}{2kT}v^2} + v^2 e^{-\frac{m}{2kT}v^2}\left[-\frac{m2v}{2kT}\right] = 0$$

$$\left[2v - \frac{mv^3}{kT}\right]e^{-\frac{m}{2kT}v^2} = 0$$

$$\left[2 - \frac{mv^2}{kT}\right]ve^{-\frac{m}{2kT}v^2} = 0$$

Three solutions: $v = 0$, $v \to \infty$, and $2 - \frac{mv^2}{kT} = 0$. The first two are minima since the function is zero at these points. We want the third solution, which is the maximum.

$$\boxed{v_p = \sqrt{\frac{2kT}{m}}}$$

HW-I3. The Average Speed. In general, show that the average speed for a particle with the Maxwell-Boltzman velocity distribution is given by

$\bar{v} = \sqrt{\frac{8kT}{\pi m}}$. **Solution.** Start with $f(v) = 4\pi\left[\frac{m}{2\pi kT}\right]^{3/2} v^2 e^{-\frac{m}{2kT}v^2}$.

$$\bar{v} = \int_0^\infty v f(v) dv = \int_0^\infty v 4\pi\left[\frac{m}{2\pi kT}\right]^{3/2} v^2 e^{-\frac{m}{2kT}v^2} dv$$

$$\bar{v} = \int_0^\infty v f(v) dv = 4\pi\left[\frac{m}{2\pi kT}\right]^{3/2} \int_0^\infty v^3 e^{-\frac{m}{2kT}v^2} dv$$

The problem says you may look up the integral. Below I went ahead and did for review.

$$\int_0^\infty x e^{-\alpha x^2} dx = -\frac{1}{2\alpha}\int_0^\infty -2\alpha x e^{-\alpha x^2} dx = -\frac{1}{2\alpha} e^{-\alpha x^2}\Big|_0^\infty = \frac{1}{2\alpha}$$, as

$$\int_0^\infty x^3 e^{-\alpha x^2} dx = -\frac{d}{d\alpha}\int_0^\infty x e^{-\alpha x^2} dx = -\frac{d}{d\alpha}\frac{1}{2\alpha} = \frac{1}{2\alpha^2}.$$

Let $\alpha = \frac{m}{2kT}$. Then,

$$\bar{v} = 4\pi\left[\frac{m}{2\pi kT}\right]^{3/2} \int_0^\infty v^3 e^{-\frac{m}{2kT}v^2} dv = 4\pi\left[\frac{m}{2\pi kT}\right]^{3/2} \frac{1}{2\alpha^2}$$

$$\bar{v} = 4\pi\left[\frac{m}{2\pi kT}\right]^{3/2} \frac{1}{2}\frac{4k^2T^2}{m^2} \qquad \bar{v} = 2\pi\left[\frac{1}{\pi}\right]^{3/2}\left[\frac{m}{2kT}\right]^{3/2} \frac{2^2k^2T^2}{m^2}$$

$$\bar{v} = 2\pi\left[\frac{1}{\pi}\right]^{3/2}\sqrt{\frac{2kT}{m}} \qquad \bar{v} = 2\sqrt{\frac{2kT}{\pi m}} \qquad \boxed{\bar{v} = \sqrt{\frac{8kT}{\pi m}}}$$

HW-I4. The Root-Mean-Square Speed.

Root meat square speed $v_{rms} = \sqrt{\overline{v^2}}$. Note $f(v) = 4\pi\left[\frac{m}{2\pi kT}\right]^{3/2} v^2 e^{-\frac{m}{2kT}v^2}$.

$$\overline{v^2} = \int_0^\infty v^2 f(v)dv = 4\pi\left[\frac{m}{2\pi kT}\right]^{3/2} \int_0^\infty v^4 e^{-\frac{m}{2kT}v^2} dv$$

The problem says you may look up the integral. Below I went ahead and obtained it from our

$$\int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \text{ and derivatives.}$$

$$\int_0^\infty x^4 e^{-\alpha x^2} dx = \left[-\frac{d}{d\alpha}\right]^2 \int_0^\infty e^{-\alpha x^2} dx = \left[-\frac{d}{d\alpha}\right]^2 \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x^4 e^{-\alpha x^2} dx = \left[-\frac{d}{d\alpha}\right]\frac{1}{2}\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\frac{1}{\alpha} = \frac{1}{2}\frac{1}{2}\frac{3}{2}\sqrt{\frac{\pi}{\alpha}}\frac{1}{\alpha^2} = \frac{3}{8}\sqrt{\frac{\pi}{\alpha}}\frac{1}{\alpha^2}$$

Let $\alpha = \frac{m}{2kT}$. Then,

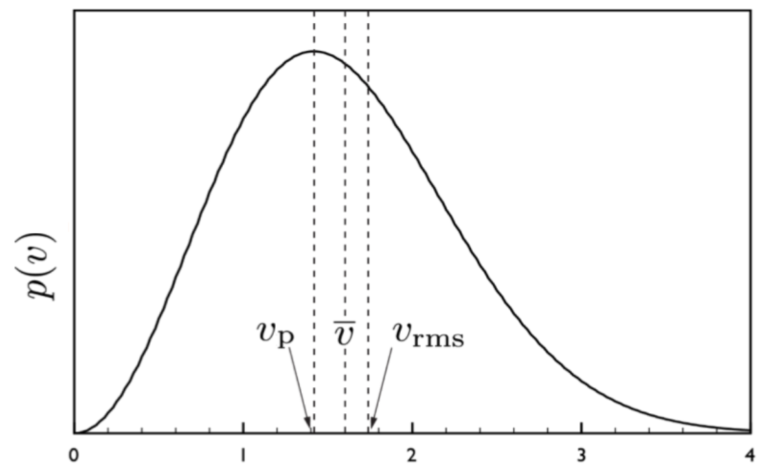
$$\overline{v^2} = 4\pi\left[\frac{m}{2\pi kT}\right]^{3/2} \int_0^\infty v^4 e^{-\frac{m}{2kT}v^2} dv = 4\pi\left[\frac{m}{2\pi kT}\right]^{3/2} \frac{3}{8}\sqrt{\frac{\pi}{\alpha}}\frac{1}{\alpha^2}$$

$$\overline{v^2} = 4\left[\frac{m}{2kT}\right]^{3/2} \frac{3}{8}\sqrt{\frac{1}{\alpha}}\frac{1}{\alpha^2} = \frac{3}{2}\left[\frac{m}{2kT}\right]^{3/2}\left[\frac{1}{\alpha}\right]^{5/2} = \frac{3}{2}\left[\frac{m}{2kT}\right]^{3/2}\left[\frac{2kT}{m}\right]^{5/2}$$

$$\overline{v^2} = \frac{3}{2}\left[\frac{m}{2kT}\right]^{3/2}\left[\frac{2kT}{m}\right]^{3/2}\frac{2kT}{m} = \frac{3}{2}\frac{2kT}{m} = \frac{3kT}{m}$$

$$v_{rms} = \sqrt{\overline{v^2}} = \sqrt{\frac{3kT}{m}} \quad \Rightarrow \quad \boxed{v_{rms} = \sqrt{\frac{3kT}{m}}}$$

HW-I5. The Big Three. Show that $v_p < \bar{v} < v_{rms}$. What do the numbers along the horizontal axis in the physics.stackexchange.com graph below represent?



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The horizontal axis is in units of $\sqrt{\dfrac{kT}{m}}$.