Theoretical Physics
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Chapter L Homework. The Dirac Equation

HW-L1. Dirac Matrices. Show that $\alpha_2\alpha_3 + \alpha_3\alpha_2 = 0$ by explicitly multiplying out the 4 x 4 matrices. Then use the shortcut method with $\alpha_2 = \begin{bmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{bmatrix}$ and $\alpha_3 = \begin{bmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{bmatrix}$ to show the same result.

HW-L2. Pauli Matrices. We have encountered the Pauli matrices on various occasions.

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

From Chapter K we have the following result.

$$\hat{n}\cdot\vec{\sigma} = \sin\theta\cos\phi \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \sin\theta\sin\phi \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Show $(\hat{n}\cdot\vec{\sigma})^{2k} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $(\hat{n}\cdot\vec{\sigma})^{2k+1} = \hat{n}\cdot\vec{\sigma}$ for $k = 0, 1, 2, \ldots$.

HW-L3. Generating SU(2) Matrices. Use the definition of a matrix A in the exponent (e^A) as given in HW-K1 to show the following with the help of your results from HW-L2.

$$e^{i\alpha\hat{n}\cdot\vec{\sigma}} = I\cos\alpha + i\hat{n}\cdot\vec{\sigma}\sin\alpha \text{ .}$$

HW-L4. Unitary Matrices. In Chapter J we saw that the matrix

$$A = \begin{bmatrix} a_r + ia_i & b_r + ib_i \\ -b_r + ib_i & a_r - ia_i \end{bmatrix}$$ is unitary if all the “a” and “b” factors are real.

Explicity write out the 2 x 2 matrix

$$U = e^{i\alpha\hat{n}\cdot\vec{\sigma}} = I\cos\alpha + i\hat{n}\cdot\vec{\sigma}\sin\alpha$$ and show that it is unitary.

Hint. If you show U has the form given above for A with real “a” and “b” factors, you got it.

HW-L5. Special Unitary Matrices. Show that $U = e^{i\alpha \hat{n}\cdot\vec{\sigma}} = I\cos\alpha + i\hat{n}\cdot\vec{\sigma}\sin\alpha$ is also a special matrix (i.e., the determinant $\det U = 1$) by explicitly calculating the determinant of the 2 x 2 matrix.

Hint. You start with U written out in 2 x 2 form as found in HW-L4.

Note: You do need to have this matrix from HW-L4 correct to get full credit for HW-L5.

Summary Comments of Your Achievement

You have found the exponential map between a Lie Algebra (described by the commutators of the Pauli matrices) and a Lie Group (all the matrices in the special unitary group). The Pauli matrices alone can generate all the $SU(2)$ matrices.

The Pauli matrices are generators of the group SU(2). They generate all the SU(2) matrices by the following exponential relationship.

$$U = e^{i\alpha \hat{n}\cdot\vec{\sigma}}$$

The three free parameters are the three angles α, θ, and ϕ. The generators satisfy a Lie Algebra:

$$\left[\sigma_j, \sigma_k\right] = 2i\varepsilon_{jkl}\,\sigma_l\,.$$

Now you have deep insight into the relationship between a Lie Algebra and its associated group.