Theoretical Physics
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Chapter M. The Method of Frobenius

HW-M1. Sines and Cosines. Solve the differential equation $y''+y=0$.

Step 1. “Series Plug In.”

$$y(x)=\sum_{k=0}^{\infty}a_k x^k \qquad y'(x)=\sum_{k=0}^{\infty}k a_k x^{k-1} \qquad y''(x)=\sum_{k=0}^{\infty}k(k-1)a_k x^{k-2}$$

Then $y''+y=0$ becomes $\sum_{k=0}^{\infty}k(k-1)a_k x^{k-2}+\sum_{k=0}^{\infty}a_k x^k=0$.

Step 2. "Fix the Exponents."

$$\sum_{k=0}^{\infty}k(k-1)a_k x^{k-2}+\sum_{k=0}^{\infty}a_k x^k=0$$

Let $m=k-2$. Then, $k=m+2$, $k-1=m+1$, and $k-2=m$.

$$\sum_{m+2=0}^{\infty}(m+2)(m+1)a_{m+2}x^m+\sum_{k=0}^{\infty}a_k x^k=0$$

Relabel m as k.

$$\sum_{k+2=0}^{\infty}(k+2)(k+1)a_{k+2}x^k+\sum_{k=0}^{\infty}a_k x^k=0$$

Can we start k out as 0? Yes since k above really starts at -2 and then -1, both giving 0.

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k+\sum_{k=0}^{\infty}a_k x^k=0$$

Step 3. “The Arbitrary Trick.” We pull out the common x^k factor, arriving at the following.

$$\sum_{k=0}^{\infty}\left[(k+2)(k+1)a_{k+2}+a_k\right]x^k=0$$

Since the equation must be true for all x and x is arbitrary, the quantity inside the brackets must vanish.

$$(k+2)(k+1)a_{k+2}+a_k=0$$

Step 4. “The Recurrence Relation.”

$$a_{k+2}=-\frac{1}{(k+2)(k+1)}a_k$$

For the even solution, we take $a_0 = 1$ and $a_1 = 0$ as instructed in the problem.

$$a_{k+2} = -\frac{1}{(k+2)(k+1)} a_k \qquad a_2 = -\frac{1}{(0+2)(0+1)} a_0 = -\frac{1}{2}$$

$$a_4 = -\frac{1}{(2+2)(2+1)} a_2 = -\left[\frac{1}{4\cdot 3}\right] a_2 = -\left[\frac{1}{4\cdot 3}\right]\left[-\frac{1}{2}\right] = \frac{1}{4!}$$

$$a_6 = -\frac{1}{(4+2)(4+1)} a_4 = -\left[\frac{1}{6\cdot 5}\right] a_5 = -\left[\frac{1}{6\cdot 5}\right]\frac{1}{4!} = -\frac{1}{6!}$$

The even series solution is $f(x) = a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \ldots$

$$\boxed{f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \cos x}$$

For the odd solution, we take $a_0 = 0$ and $a_1 = 1$ as instructed in the problem.

$$a_{k+2} = -\frac{1}{(k+2)(k+1)} a_k \qquad a_3 = -\frac{1}{(1+2)(1+1)} a_1 = -\frac{1}{3\cdot 2}$$

$$a_5 = -\frac{1}{(3+2)(3+1)} a_3 = -\left[\frac{1}{5\cdot 4}\right] a_3 = -\left[\frac{1}{5\cdot 4}\right]\left[-\frac{1}{3\cdot 2}\right] = \frac{1}{5!}$$

$$a_7 = -\frac{1}{(5+2)(5+1)} a_5 = -\left[\frac{1}{7\cdot 6}\right] a_5 = -\left[\frac{1}{7\cdot 6}\right]\frac{1}{5!} = -\frac{1}{7!}$$

The odd series solution is $g(x) = a_1 + a_3 x^3 + a_5 x^5 + a_7 x^7 + \ldots$

$$\boxed{g(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sin x}$$

HW-M2. The Laguerre Differential Equation.

Edmund Laguerre (1834-1886)
Courtesy School of Mathematics & Statistics
University of St. Andrews, Scotland

The Laguerre differential equation is

$$xy'' + (1-x)y' + ny = 0,$$

where $n = 0, 1, 2, 3, \ldots$

Step 1. "Series Plug In."

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$y' = \sum_{k=0}^{\infty} k a_k x^{k-1} \quad \text{and} \quad y''(x) = \sum_{k=0}^{\infty} k(k-1) a_k x^{k-2}$$

$$x\sum_{k=0}^{\infty} k(k-1) a_k x^{k-2} + (1-x)\sum_{k=0}^{\infty} k a_k x^{k-1} + n\sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty} k(k-1) a_k x^{k-1} + \sum_{k=0}^{\infty} k a_k x^{k-1} - \sum_{k=0}^{\infty} k a_k x^k + n\sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty} k^2 a_k x^{k-1} - \sum_{k=0}^{\infty} k a_k x^k + n\sum_{k=0}^{\infty} a_k x^k = 0$$

Step 2. "Fix the Exponents." Let $m = k-1$, $k = m+1$.

$$\sum_{m=-1}^{\infty} (m+1)^2 a_{m+1} x^m - \sum_{k=0}^{\infty} k a_k x^k + n\sum_{k=0}^{\infty} a_k x^k = 0$$

Relabel and check that to start at k = 0 is okay. We can since m = -1 gives nothing above.

$$\sum_{k=0}^{\infty}\left[(k+1)^2 a_{k+1}-(k-n)a_k\right]x^k=0$$

Step 3. "The Arbitrary Trick

$$(k+1)^2 a_{k+1}-(k-n)a_k=0$$

Step 4. "The Recurrence Relation."

$$\boxed{a_{k+1}=\frac{(k-n)}{(k+1)^2}a_k}$$

We choose $a_0=n!$. A alternative convention is to choose the zeroth term to be 1.

HW-M3. Laguerre Polynomials.

0. The Zeroth Laguerre Polynomial ($n=0$)

$$a_{k+1}=\frac{(k-0)}{(k+1)^2}a_k$$

$$a_1=a_{0+1}=\frac{(0-0)}{(0+1)^2}a_0=0$$ so all we have is the zeroth coefficient.

$$L_0(x)=a_0=0!$$ since we always choose $a_0=n!$.

$$\boxed{L_0(x)=1}$$

1. The First Laguerre Polynomial ($n=1$)

$$a_{k+1}=\frac{(k-1)}{(k+1)^2}a_k$$

$$a_1=a_{0+1}=\frac{(0-1)}{(0+1)^2}a_0=-a_0$$

$$a_2 = a_{1+1} = \frac{(1-1)}{(1+1)^2} a_1 = 0$$

$$L_1(x) = a_0 + a_1 x$$

Choose $a_0 = n! = 1! = 1$

$$\boxed{L_1(x) = 1 - x}$$

2. The Second Laguerre Polynomial ($n = 2$)

$$a_{k+1} = \frac{(k-2)}{(k+1)^2} a_k$$

$$a_1 = a_{0+1} = \frac{(0-2)}{(0+1)^2} a_0 = -2a_0$$

$$a_2 = a_{1+1} = \frac{(1-2)}{(1+1)^2} a_0 = -\frac{1}{4} a_1 = -\frac{1}{4}(-2a_0) = \frac{1}{2} a_0$$

$$a_3 = a_{2+1} = \frac{(2-2)}{(2+1)^2} a_2 = 0$$

$$L_2(x) = a_0 + a_1 x + a_2 x^2$$

$$a_0 = n! = 2! = 2$$

$$a_1 = -2a_0 = -4$$

$$a_2 = \frac{1}{2} a_0 = 1$$

$$L_2(x) = 2 - 4x + x^2$$

$$\boxed{L_2(x) = x^2 - 4x + 2}$$

3. The Third Laguerre Polynomial ($n = 3$ **)**

$$a_{k+1} = \frac{(k-3)}{(k+1)^2} a_k \quad \text{We pick } a_0 = 3! = 6$$

$$a_1 = a_{0+1} = \frac{(0-3)}{(0+1)^2} a_0 = -3a_0 = -18$$

$$a_2 = a_{1+1} = \frac{(1-3)}{(1+1)^2} a_1 = -\frac{2}{4} a_1 = -\frac{1}{2} a_1 = -\frac{1}{2}(-18) = 9$$

$$a_3 = a_{2+1} = \frac{(2-3)}{(2+1)^2} a_2 = -\frac{1}{9} a_2 = -\frac{1}{9}(9) = -1$$

$$L_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$L_3(x) = 6 - 18x + 9x^2 - x^3$$

$$\boxed{L_3(x) = -x^3 + 9x^2 - 18x + 6}$$

I will throw in the fourth Laguerre Polynomial (NOT REQUIRED IN THE HOMEWORK).

4. The Fourth Laguerre Polynomial ($n = 4$ **)**

$$a_{k+1} = \frac{(k-3)}{(k+1)^2} a_k \quad \text{We pick } a_0 = 4! = 24$$

$$a_1 = a_{0+1} = \frac{(0-4)}{(0+1)^2} a_0 = -4a_0 = -96$$

$$a_2 = a_{1+1} = \frac{(1-4)}{(1+1)^2} a_1 = -\frac{3}{4} a_1 = -\frac{3}{4}(-96) = 72$$

$$a_3 = a_{2+1} = \frac{(2-4)}{(2+1)^2} a_2 = -\frac{2}{9} a_2 = -\frac{2}{9}(72) = -16$$

$$a_4 = a_{3+1} = \frac{(3-4)}{(3+1)^2} a_3 = -\frac{1}{16} a_3 = -\frac{1}{16}(-16) = 1$$

$$L_4(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$L_4(x) = 24 - 96x + 72x^2 - 16x^3 + x^4$$

$$\boxed{L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 24}$$