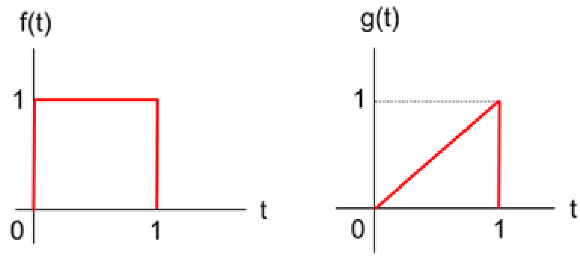
Theoretical Physics
Prof. Ruiz, UNC Asheville
Chapter Q Homework Solutions. Laplace Transforms.

HW-Q1. Laplace Transform. Find the Laplace transform F(s) for the square pulse f(t) shown below by explicitly doing the Laplace transform integral. Then use the "derivative trick" for integration to obtain the Laplace transform G(s) of the ramp pulse g(t) from your result F(s) for the square pulse.



Finally, give F(1) and G(1) in terms of e, where e is the natural base.

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^1 e^{-st}dt$$

$$F(s) = \left.\frac{e^{-st}}{-s}\right|_0^1 = \left[\frac{e^{-s}}{-s}\right] - \left[\frac{1}{-s}\right] = \frac{1}{s}\left[1 - e^{-s}\right]$$

$$\boxed{F(s) = \frac{1}{s}\left[1 - e^{-s}\right]}$$

$$G(s) = \int_0^\infty g(t)e^{-st}dt = \int_0^1 te^{-st}dt$$

$$G(s) = -\frac{d}{ds}\int_0^1 e^{-st}dt = -\frac{dF(s)}{ds}$$

$$G(s) = -\frac{d}{ds}\left[\frac{1}{s}(1 - e^{-s})\right]$$

$$G(s) = \frac{1}{s^2}(1 - e^{-s}) - \frac{1}{s}(+e^{-s})$$

$$G(s) = \frac{1}{s^2}(1 - e^{-s}) - \frac{1}{s}e^{-s}$$

Summary: $F(s) = \frac{1}{s}\left[1 - e^{-s}\right]$ $G(s) = \frac{1}{s^2}(1 - e^{-s}) - \frac{1}{s}e^{-s}$

$$F(1) = \frac{1}{1}\left[1 - e^{-1}\right] = 1 - \frac{1}{e}$$

$$G(1) = \frac{1}{1}(1 - e^{-1}) - \frac{1}{1}e^{-1} = 1 - \frac{2}{e}$$

HW-Q2. Laplace Transform Shift Property. Calculate the Laplace transform $G(s)$ for $g(t) = t^n e^{-bt}$ two ways as described below, where $b > 0$.

a) Do the integral for the Laplace transform using the derivative trick.

b) Use the shifting property: if $g(t) = f(t)e^{-\alpha t}$, then $G(s) = F(s - a)$, $s > a$.

Method a.

$$G(s) = \int_0^\infty t^n e^{-bt} e^{-st} dt$$

$$G(s) = \left[-\frac{d}{ds}\right]^n \int_0^\infty e^{-(s+b)t} dt = \left[-\frac{d}{ds}\right]^n L\{e^{-bt}\}$$

$$G(s) = \left[-\frac{d}{ds}\right]^n \left[\frac{1}{s+b}\right]$$

$$G(s) = \frac{n!}{(s+b)^{n+1}}$$

Method b.

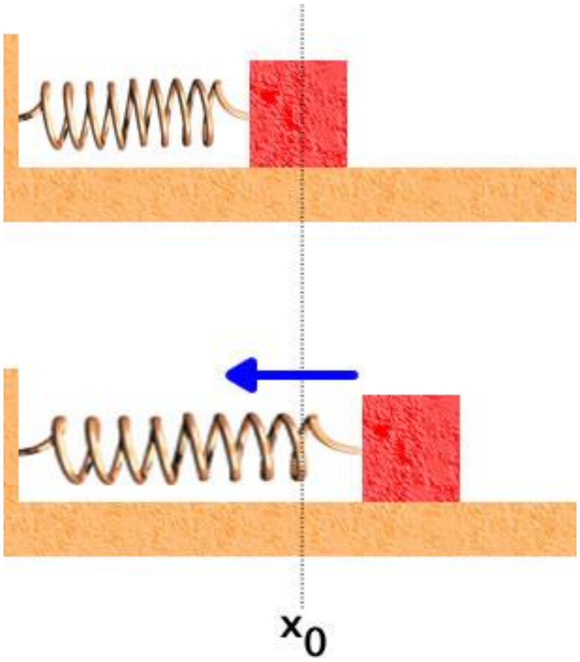
$g(t) = f(t)e^{\alpha t}$, then $G(s) = F(s-a)$

Take $f(t) = t^n$. Therefore $F(s) = \dfrac{n!}{s^{n+1}}$.

For $g(t) = f(t)e^{-bt}$, we have $G(s) = F(s+b)$

$$G(s) = \frac{n!}{(s+b)^{n+1}}$$

HW-Q3. Solving a Differential Equation.



Courtesy David M. Harrison
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Use Laplace transforms to solve the differential equation

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$$

where you smack the block initially so that

$$x(0)=0 \quad \text{and} \quad v(0)=A\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}.$$

Simply your math using these definitions:

$$\omega_0=\sqrt{\frac{k}{m}},\ \beta=\frac{b}{2m},\ \omega^2=\omega_0^2-\beta^2.$$

We will need the Laplace transform for the first and second derivatives:

$$L\{f'(t)\}=sF(s)-f(0) \quad \text{and} \quad L\{f''(t)\}=s^2F(s)-sf(0)-f'(0).$$

1. Take the Laplace Transform

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$$

$$m\left[s^2F(s)-sx(0)-v(0)\right]+b\left[sF(s)-x(0)\right]+kF(s)=0$$

With our initial conditions $x(0)=0$ and $v(0)=A\omega$ we have

$$m\left[s^2F(s)+A\omega\right]+bsF(s)+kF(s)=0$$

2. Solve Your Algebraic equation

$$F(s)\left[ms^2 + bs + k\right] = mA\omega$$

$$F(s) = A\frac{\omega}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$

$$F(s) = A\frac{\omega}{s^2 + 2\beta s + \omega_0^2}$$

Complete the square.

$$F(s) = A\frac{\omega}{(s+\beta)^2 + \omega_0^2 - \beta^2}$$

$$F(s) = A\frac{\omega}{(s+\beta)^2 + \omega^2}$$

3. Use the Laplace Transform Table to Get Your Solution

$$x(t) = L^{-1}\{F(s)\} = L^{-1}\{A\frac{\omega}{(s+\beta)^2 + \omega^2}\}$$

$$x(t) = AL^{-1}\{\frac{\omega}{(s+\beta)^2 + \omega^2}\}$$

$$x(t) = Ae^{-\beta t}\sin\omega t$$