Theoretical Physics
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Chapter R Homework. Convolution

R1. Convolution. You will work with the following two functions. $f(t)=t$ and $g(t)=t^2$

a. Find the convolution $f(t)*g(t)=\int_0^t f(u)g(t-u)du$.

b. Show that if you calculate the convolution by the following formula, you get the same result. Since these formulas are true in general, the convolution operation is commutative.

$$g(t)*f(t)=\int_0^t g(u)f(t-u)du$$

c. Show for this case that the Laplace transform of the convolution is given by the product of the Laplace transforms of each function. When you do this part, you can look up the Laplace transforms in our "Laplace Transform Table" found in our text.

$$L\{g(t)*f(t)\}=L\{g(t)\}L\{f(t)\}=G(s)F(s)$$

R2. Dumping Radioactive Waste. There is a pristine dumping ground totally free of any kind of trash. Then in comes the dump trucks dumping radioactive waste according to the dumping function $d(t)$. The radioactive-decay differential equation must now be modified to include dumping.

$$\frac{dn(t)}{dt}=-\lambda n(t)+d(t)$$

Your first term on the right is your radioactive-loss due to decay. That piece is still proportional to $n(t)$. But you have to add the gain in radioactive particles $d(t)$ to get your total rate of change $\frac{dn(t)}{dt}$. Show that the Laplace transform of your differential equation leads to your solution in the form $N(s)=F(s)D(s)$ where $L\{n(t)\}=N(s)$, $L\{d(t)\}=D(s)$, and $F(s)$ is everything else. What is $f(t)$? Give the solution $n(t)$ in the form of a convolution. Now consider a dumping function $d(t)=1-t$ for $0\le t\le 1$ and $d(t)=0$ elsewhere. Sketch this dumping function. Show that the "derivative trick" can be used to express your solution for this particular dumping function for times $1\le t$ as

$$n(t)=e^{-\lambda t}\left[1-\frac{d}{d\lambda}\right]\left[\frac{1}{\lambda}(e^{\lambda}-1)\right].$$

Finally evaluate the above and express $n(t)$ in simplest terms.