Theoretical Physics
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Chapter U Homework - Solutions. Green's Functions

U1. The Green's Function for Radioactive Decay. The differential equation for radioactive decay is $\frac{dn(t)}{dt} + \lambda n(t) = 0$, which comes from $\frac{dn(t)}{dt} = -\lambda n(t)$. Use your four-step procedure (delta function, Fourier transform, complex integration, Green's function) to show that the Green's function for the radioactive-decay differential equation is $G(t,0) = e^{-\lambda t}$.

Solution.

Step 1. Delta Function.

$$\frac{dn(t)}{dt} + \lambda n(t) = \delta(t)$$

Step 2. Fourier Transform.

$$\Im\{\frac{dn(t)}{dt} + \lambda n(t)\} = \Im\{\delta(t)\}$$

$$\Im\{\frac{dn(t)}{dt}\} + \lambda \Im\{n(t)\} = \Im\{\delta(t)\}$$

$$i\omega N(\omega) + \lambda N(\omega) = \frac{1}{\sqrt{2\pi}}$$

$$N(\omega)(i\omega + \lambda) = \frac{1}{\sqrt{2\pi}}$$

$$N(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{i\omega + \lambda}$$

Summary: We went to Fourier transform space and solved the problem algebraically.

Step 3. Complex Integration. We need to get back to regular space. So we apply an inverse Fourier transform to get back.

$$n(t) = \Im^{-1}\{N(\omega)\} \equiv G(t,0)$$

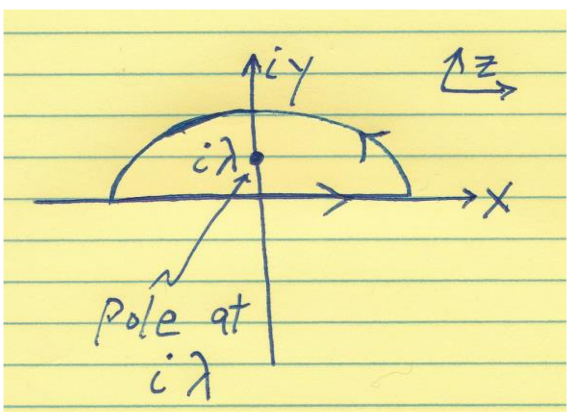
$$G(t,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} N(\omega) e^{i\omega t} d\omega$$

Now use $N(\omega) = \dfrac{1}{\sqrt{2\pi}} \dfrac{1}{i\omega + \lambda}$ from Step 2.

$$G(t,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \frac{1}{i\omega + \lambda} \right] e^{i\omega t} d\omega$$

Here is where the complex integration kicks in.

$$G(t,0) = \frac{-i}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1}{\omega - i\lambda} \right] e^{i\omega t} d\omega \rightarrow \frac{-i}{2\pi} \oint \frac{e^{izt}}{z - i\lambda} dz$$



There is a pole at $z = i\lambda$. Note that the e^{izt} factor means we need to close in the upper plane so that

$$\lim_{y \to \infty} e^{izt} = \lim_{y \to \infty} e^{i(x+iy)t} = e^{ixt} \lim_{y \to \infty} e^{-yt} = 0$$

$$G(t,0) = \frac{-i}{2\pi} \oint \frac{e^{izt}}{z - i\lambda} dz$$

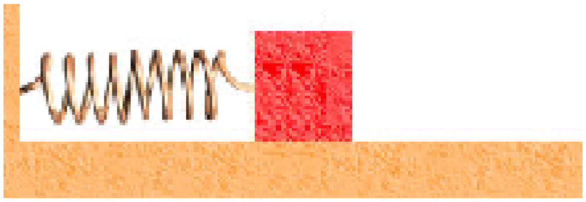
$$G(t,0) = \frac{-i}{2\pi} \left[2\pi i Res\left(\frac{e^{izt}}{z - i\lambda}, i\lambda\right) \right]$$

$$G(t,0) = e^{izt} \Big|_{z=i\lambda}$$

Step 4. Green's Function.

$$\boxed{n(t) = G(t,0) = e^{-\lambda t}}$$

U2. The Green's Function for the Damped Harmonic Oscillator.



Courtesy David M. Harrison
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The differential equation for the damped harmonic oscillator is

$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = 0$, which comes from $F = -kx - bv = ma$ with

$\omega_0^2 = \frac{k}{m}$ and $\beta = \frac{b}{2m}$. For your specific problem $\alpha^2 = \omega_0^2 - \beta^2 > 0$.

Use your four-step procedure (delta function, Fourier transform, complex integration, Green's function) to show that the Green's function for the damped harmonic oscillator system is given by

$$G(t,0) = \frac{1}{\alpha}e^{-\beta t}\sin(\alpha t).$$

Solution.

Step 1. Delta Function.

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = \delta(t)$$

Step 2. Fourier Transform.

$$\Im\{\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x\} = \Im\{\delta(t)\}$$

$$\Im\{\frac{d^2x}{dt^2}\} + 2\beta\Im\{\frac{dx}{dt}\} + \Im\{\omega_0^2 x\} = \Im\{\delta(t)\}$$

$$-\omega^2 X(\omega) + 2\beta i\omega X(\omega) + \omega_0^2 X(\omega) = \frac{1}{\sqrt{2\pi}}$$

Solve the algebraic equation in the transformed space.

$$X(\omega)\left[-\omega^2+2\beta i\omega+\omega_0^2\right]=\frac{1}{\sqrt{2\pi}}$$

$$X(\omega)=\frac{1}{\sqrt{2\pi}}\frac{1}{(-\omega^2+2\beta i\omega+\omega_0^2)}$$

Step 3. Complex Integration. We need to get back to regular space. So we apply an inverse Fourier transform to get back.

$$x(t)=G(t,0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{(-\omega^2+2\beta i\omega+\omega_0^2)}d\omega$$

$$G(t,0)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\omega^2-2\beta i\omega-\omega_0^2}d\omega$$

We will go to the complex plane and close upward since we have $e^{i\omega t}$.

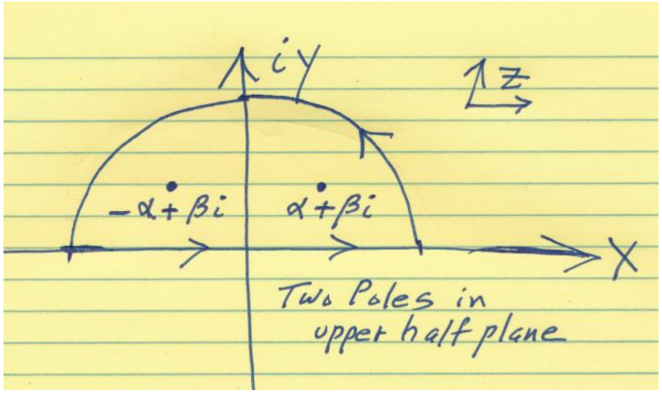
$$G(t,0)=-\frac{1}{2\pi}\oint\frac{e^{izt}}{z^2-2\beta iz-\omega_0^2}dz$$

$$z^2-2\beta iz-\omega_0^2=0$$

$$\frac{-(-2\beta i)\pm\sqrt{(-2\beta i)^2-4(1)(-\omega_0^2)}}{2(1)}$$

$$\frac{2\beta i\pm\sqrt{-4\beta^2+4\omega_0^2}}{2} \quad => \quad \frac{2\beta i\pm 2\sqrt{\omega_0^2-\beta^2}}{2}$$

$\beta i\pm\sqrt{\omega_0^2-\beta^2}$ => $\beta i\pm\alpha$ since it was given that $\alpha^2=\omega_0^2-\beta^2>0$



$$G(t,0) = -\frac{1}{2\pi} 2\pi i \sum_n Res(F, z_n)$$

$$F(z) = \frac{e^{izt}}{(z - z_1)(z - z_2)}$$ with

$$z_1 = \beta i + \alpha$$ and $$z_2 = \beta i - \alpha$$

$$G(t,0) = -i\left[Res(F, z_1) + Res(F, z_2)\right]$$

$$G(t,0) = -i\left[\frac{e^{iz_1 t}}{(z_1 - z_2)} + \frac{e^{iz_2 t}}{(z_2 - z_1)}\right]$$

$$G(t,0) = \frac{-i}{z_1 - z_2}\left[e^{iz_1 t} - e^{iz_2 t}\right]$$

$$G(t,0) = \frac{-i}{2\alpha}\left[e^{i(\beta i + \alpha)t} - e^{i(\beta i - \alpha)t}\right]$$

$$G(t,0) = \frac{-i}{2\alpha} e^{-\beta t}\left[e^{i\alpha t} - e^{-i\alpha t}\right]$$

$$G(t,0) = \frac{1}{2i\alpha} e^{-\beta t}\left[e^{i\alpha t} - e^{-i\alpha t}\right]$$

$$G(t,0) = \frac{1}{\alpha} e^{-\beta t}\left[\frac{e^{i\alpha t} - e^{-i\alpha t}}{2i}\right]$$

$$\boxed{G(t,0) = \frac{1}{\alpha} e^{-\beta t} \sin(\alpha t)}$$