

1. Statistical Mechanics. There are $N = 26,000,000$ particles, where some are in energy state $E_1 = 0$ and the rest are in state $E_2 = \varepsilon$. Give the occupation number n_1 for the 1st state when

$$\varepsilon = kT, \text{ using the approximation } e = 2.718\dots \approx 2\frac{5}{7} = \frac{19}{7}.$$

2. Eigenvalues. Find the eigenvalues and eigenvectors for σ_y .

3. Commutator. In class we showed that $[x, p]\psi = i\hbar\psi$, i.e., $[x, p] = i\hbar$.

Express the commutator $\left[x^2, x \frac{d}{dx} \right]$ in simplest terms.

4. Dirac Four-Spinor. When a spin-1/2 is moving, the Dirac four spinor for the particle is given by

$$u(\vec{p}, s) = \sqrt{\frac{E + mc^2}{2E}} \begin{bmatrix} \phi(s) \\ \frac{c \vec{\sigma} \cdot \vec{p}}{E + mc^2} \phi(s) \end{bmatrix}, \text{ where } \phi(s) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Express this four-spinor in simplest terms for the case where

$$E = 2, mc^2 = 1, c\vec{p} = \hat{i} + \hat{j} + \hat{k}, \text{ and } \phi(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

1. **Statistical Mechanics.** $E_1 = 0$, $E_2 = \varepsilon$, $\varepsilon = kT$, $N = 26,000,000$, $e \approx 19/7$.

$$Z = \sum_i e^{-\frac{E_i}{kT}} = 1 + e^{-\frac{\varepsilon}{kT}} = 1 + e^{-1} = 1 + \frac{7}{19} = \frac{26}{19}$$

$$n_1 = Ne^{-\frac{E_1}{kT}} / Z = N / Z = 26,000,000 [19 / 26] = 19,000,000$$

2. **Eigenvalues.**

$$\sigma_y \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \lambda \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \lambda \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\lambda & -i \\ i & -\lambda \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$\det \begin{bmatrix} -\lambda & -i \\ i & -\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \quad \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -ic_2 \\ ic_1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$c_2 = ic_1 \Rightarrow u = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ normalized. For } c_2 = -ic_1, v = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

3. **Commutator.**

$$\left[x^2, x \frac{d}{dx} \right] \psi = x^2 \left(x \frac{d\psi}{dx} \right) - x \frac{d}{dx} (x^2 \psi) = x^3 \frac{d\psi}{dx} - x(2x\psi) - x^3 \frac{d\psi}{dx}$$

$$\left[x^2, x \frac{d}{dx} \right] \psi = -x(2x\psi). \text{ Therefore, } \left[x^2, x \frac{d}{dx} \right] = -2x^2.$$

4. **Dirac Four-Spinor.** $E = 2$, $mc^2 = 1$, $\vec{c}\vec{p} = \hat{i} + \hat{j} + \hat{k}$, and $\phi(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$u(\vec{p}, s) = \sqrt{\frac{E + mc^2}{2E}} \begin{bmatrix} \phi(s) \\ \frac{\vec{c}\vec{\sigma} \cdot \vec{p}}{E + mc^2} \phi(s) \end{bmatrix} \quad \sqrt{\frac{E + mc^2}{2E}} = \sqrt{\frac{2+1}{2(2)}} = \frac{\sqrt{3}}{2}$$

$$\vec{c}\vec{\sigma} \cdot \vec{p} = \sigma_x + \sigma_y + \sigma_z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$\frac{\vec{c}\vec{\sigma} \cdot \vec{p}}{E + mc^2} \phi(s) = \frac{1}{3} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1+i \end{bmatrix} \quad \text{and} \quad u(\vec{p}, s) = \frac{\sqrt{3}}{2} \begin{bmatrix} 1 \\ 0 \\ 1/3 \\ (1+i)/3 \end{bmatrix}$$