Theoretical Physics (Prof. Ruiz/UNCA) **Exam 2** **March 22, 2021**

Open Book/Notes/Internet (90 minutes including scan and email return).

Posted Syllabus Exam Time is 6:00 pm – 7:30 pm.

Exam emailed 5:45 pm and you can start as soon as you get it.

All work and related steps must be explicitly shown for full credit.

[25] 1. Combinatorics. There are 10 students taking a flexible course where 3 of the 10 are attending face to face, another 3 are attending online, and the remaining 4 are attending in a mixed more. How many different ways can this arrangement be realized?

[25] 2. Inverse Matrix. Find the inverse matrix for the Pauli combination

$$M = \sigma_y + \sigma_z = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix}.$$

Express your final answer in terms of M .

[25] 3. Matrix as Exponent. Find the simplest form for e^A, where $A = i\theta\sigma_z$ and σ_z is the third Pauli matrix. Simplest form is defined by the least amount of characters needed to express the final answer. As an example, the function $\cos\theta$ consists of 4 characters. A left or right parenthesis counts as one character.

[25] 4. Recurrence Relation. You are given the recurrence relation

$$a_{k+2} = \frac{2(k-n)}{(k+1)(k+2)} a_k \text{ for } f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots,$$

where $k = 0,1,2,3\ldots$ and $n = 0,1,2,3\ldots$

Find the specific polynomial $f(x)$ where $a_0 = 0$, $a_1 = 1$, and $n = 5$.

[15] 1. Combinatorics. $\frac{10!}{3!3!4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{(3 \cdot 2)(3 \cdot 2)} = 10 \cdot 3 \cdot 4 \cdot 7 \cdot 5 = 4200$

[20] 2. Inverse Matrix. $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix}$. Then $M^{-1} = \frac{1}{|M|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$$M^{-1} = \frac{1}{[-1-(-i^2)]}\begin{bmatrix} -1 & i \\ -i & 1 \end{bmatrix} = \frac{1}{(-2)}\begin{bmatrix} -1 & i \\ -i & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix} = \frac{1}{2}M$$

[25] 3. Matrix as Exponent.

$$e^A = e^{i\theta\sigma_z} = I + i\theta\sigma_z - \frac{\theta^2}{2!}\sigma_z^2 - i\frac{\theta^3}{3!}\sigma_z^3 + \frac{\theta^4}{4!}\sigma_z^4 + i\frac{\theta^5}{5!}\sigma_z^5 \ldots$$

Note that $\sigma_z^2 = I$, $\sigma_z^3 = \sigma_z$, … so that even powers give I and odd powers σ_z.

$$e^A = \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right] I + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right]\sigma_z = I\cos\theta + i\sigma_z\sin\theta$$

$$e^A = \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix} + \begin{bmatrix} i\sin\theta & 0 \\ 0 & -i\sin\theta \end{bmatrix} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

[25] 4. Recurrence Relation. $a_{k+2} = \frac{2(k-n)}{(k+1)(k+2)} a_k$ with $a_0 = 0$, $a_1 = 1$, $n = 5$.

Will only have odd cases. $a_3 = \frac{2(1-5)}{(1+1)(1+2)} a_1 = -\frac{2 \cdot 4}{2 \cdot 3} a_1 = -\frac{4}{3} a_1 = -\frac{4}{3}$

$a_5 = \frac{2(3-5)}{(3+1)(3+2)} a_3 = -\frac{2 \cdot 2}{4 \cdot 5} a_3 = -\frac{1}{5} a_3 = -\frac{1}{5}(-\frac{4}{3}) = \frac{4}{15}$. Note $a_7 = 0$.

$$f(x) = x - \frac{4}{3}x^3 + \frac{4}{15}x^5$$