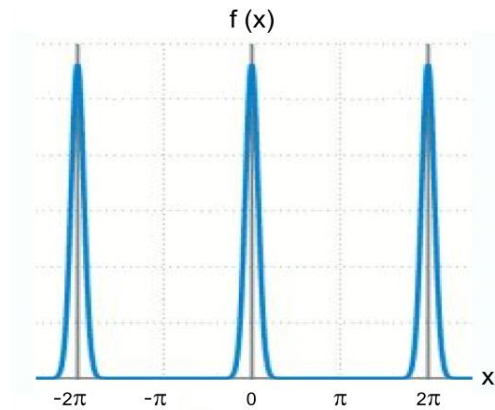


[15] Q1. **Fourier Series.** Find the Fourier Series for periodic wave $f(x)$ shown at the right, where you take the peaks to be delta functions. Write out the first 5 terms of the Fourier series in their simplest forms.



the
take the
terms of

[10] Q2. **Fourier Transform.** Now remove all the functions in the figure of Q1 except for the one at the Fourier transform $F(k)$ for your delta function at 2π . Find $F(1)$, i.e., when $k = 1$.

delta
 2π . Find
 2π .

[15] Q3. **Laplace Transform.** Use the derivative trick

integrals to show that the Laplace transform of t is $\frac{1}{s^2}$, i.e., $L\{t\} = F(s) = \frac{1}{s^2}$. You can use this result in Q4 below.

with

[15] Q4. **Laplace Transform of a Differential Equation.** A car starts from rest at $x(0) = x_0$ and accelerates with constant acceleration $a > 0$. The differential equation that describes this scenario is

$\frac{dx}{dt} = at$. Take the Laplace transform of the differential equation and obtain the Laplace transform of $x(t)$, which you can call $X(s)$. Now find $x(t)$ from the Laplace Table, where you find $L\{t^n\} = \frac{n!}{s^{n+1}}$. Use this table information to obtain the solution $x(t)$.

[15] Q5. **Convolution.** Show that the “derivative trick” can be used to express the convolution of a radioactive-waste dumping function $f(t) = t$ and impulse radioactive-decay response $g(t) = e^{-\lambda t}$

$$\text{as } f(t) * g(t) = e^{-\lambda t} \frac{d}{d\lambda} \left[\frac{e^{\lambda t} - 1}{\lambda} \right].$$

[10] Q6. **Cauchy-Riemann.** Show that $z^2 = (x + iy)^2$ is analytic.

[20] Q7. **Complex Integration.** Using complex integration, evaluate the definite integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^4 + 4}. \text{ If you have trouble, you may for half credit evaluate } I = \int_{-\infty}^{\infty} \frac{dx}{x^2 + 4} \text{ instead.}$$

1. **Fourier Series.** Dirac delta impulse train. Periodic $f(x) = \delta(x)$. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{+\pi} \delta(x) dx = \frac{1}{\pi} \quad a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} \delta(x) \cos(nx) dx = \frac{1}{\pi} \cos(nx) \Big|_{x=0} = \frac{1}{\pi} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} \delta(x) \sin(nx) dx = 0$$

$$f(x) = \frac{1}{2\pi} + \frac{1}{\pi} [\cos x + \cos(2x) + \cos(3x) + \cos(4x) \dots]$$

2. **Fourier Transform.** $F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \delta(x - 2\pi) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} e^{-ikx} \Big|_{x=2\pi} = \frac{1}{\sqrt{2\pi}} e^{-ik2\pi}$

$$F(1) = \frac{1}{\sqrt{2\pi}} e^{-i2\pi} = \frac{1}{\sqrt{2\pi}} [\cos(-2\pi) + i \sin(-2\pi)] = \frac{1}{\sqrt{2\pi}} \cos(2\pi) = \frac{1}{\sqrt{2\pi}}$$

3. **Laplace Transform.** $L\{t\} = \int_0^{\infty} t e^{-st} dt = -\frac{d}{ds} \int_0^{\infty} e^{-st} dt = -\frac{d}{ds} \left[\frac{e^{-st}}{-s} \Big|_0^{\infty} \right] = -\frac{d}{ds} \left[\frac{0-1}{-s} \right] = -\frac{d}{ds} \frac{1}{s} = \frac{1}{s^2}$

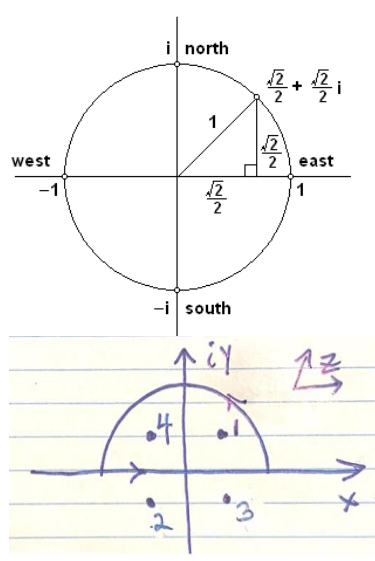
4. **Laplace Transform of a Differential Equation.** $\frac{dx}{dt} = at$ with $x(0) = x_0$ gives $sX(s) - x(0) = a \frac{1}{s^2}$

$$X(s) = x_0 \frac{1}{s} + a \frac{1}{s^3}. \text{ Then } L\{t^n\} = \frac{n!}{s^{n+1}} \Rightarrow L\{t^0\} = \frac{1}{s} = L\{1\} \text{ and } L\{t^2\} = \frac{2}{s^3} \Rightarrow x(t) = x_0 + a \frac{t^2}{2}$$

5. **Convolution.** $\int_0^t u e^{-\lambda(t-u)} du = e^{-\lambda t} \int_0^t u e^{\lambda u} du = e^{-\lambda t} \frac{d}{d\lambda} \int_0^t e^{\lambda u} du = e^{-\lambda t} \frac{d}{d\lambda} \left[\frac{e^{\lambda u}}{\lambda} \Big|_0^t \right] = e^{-\lambda t} \frac{d}{d\lambda} \left[\frac{e^{\lambda t} - 1}{\lambda} \right]$

6. **Cauchy-Riemann.** $z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi = u + iv$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y$

7. **Complex Integration.** $I = \int_{-\infty}^{\infty} \frac{dx}{x^4 + 4}$ $I = \oint \frac{1}{z^4 + 4} dz$ $I = \oint \frac{1}{(z^2 + 2i)(z^2 - 2i)} dz$



Poles: $z = \pm\sqrt{-2i}$ and $z = \pm\sqrt{2i}$. Since i is a 90° counterclockwise rotation in the complex plane, \sqrt{i} is at $45^\circ \Rightarrow \sqrt{i} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$. Since $-i$ is a 90° clockwise rotation, $\sqrt{-i}$ is at $-45^\circ \Rightarrow \sqrt{-i} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$. Then $\pm\sqrt{2i} = \pm(1+i)$ and $\pm\sqrt{-2i} = \pm(1-i)$.

Poles: $z_1 = 1+i$, $z_2 = -1-i$, $z_3 = 1-i$, $z_4 = -1+i$. Only Poles 1 and 4 are inside.

$$I = \oint \frac{1}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)} dz = \oint F(z) dz$$

$$F(z) = \frac{1}{(z - 1 - i)(z + 1 + i)(z - 1 + i)(z + 1 - i)}$$

$$Res(F, z_1) = \frac{1}{(1+i+1+i)(1+i-1+i)(1+i+1-i)} = \frac{1}{(2+2i)(2i)(2)} = \frac{1}{8i(1+i)}$$

$$Res(F, z_4) = \frac{1}{(-1+i-1-i)(-1+i+1+i)(-1+i-1+i)} = \frac{1}{(-2)(2i)(-2+2i)} = \frac{1}{8i(1-i)}$$

$$I = 2\pi i \sum_{n=1,4} Res(F, z_n) = 2\pi i \left[\frac{1}{8i(1+i)} + \frac{1}{8i(1-i)} \right] = \frac{\pi}{4} \left[\frac{1}{1+i} + \frac{1}{1-i} \right] = \frac{\pi}{4} \left[\frac{1-i+1+i}{(1+i)(1-i)} \right] = \frac{\pi}{4} \frac{2}{(1+1)} = \frac{\pi}{4}$$