Theoretical Physics **Final Exam** **December 6, 2011**
Prof. Ruiz, UNC Asheville **Sample with Solutions**

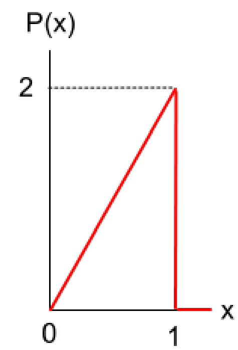
[5] 1. Groups and Matrices. Find the third matrix M_3 to go with the following to form a group where the binary operation is matrix multiplication.

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

[10] 2. Relativity and Expanding. The relativistic velocity addition formula is $\beta = \dfrac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}$, where $\beta_1 = \dfrac{v_1}{c}$, $\beta_2 = \dfrac{v_2}{c}$, and $\beta = \dfrac{v}{c}$ (your relativistic sum of velocities 1 and 2). Give β to order $\dfrac{1}{c^3}$ when $\beta_1 = \dfrac{1}{10}$ and $\beta_2 = \dfrac{1}{10}$. What is the exact answer without doing the expansion to order $1/c^3$? Give your final answers as reduced fractions.

[5] 3. Statistics.

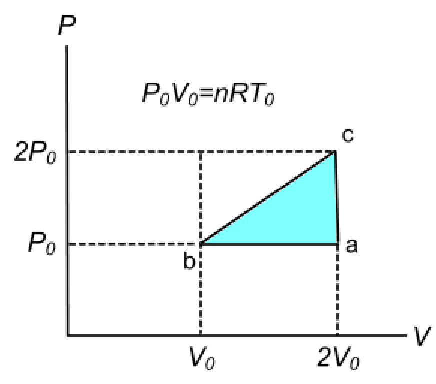
Find the average value of x for the probability distribution at the right. The function P(x) is zero when x > 1.



[15] 4. Engine. The engine illustrated has the cycle a-b-c. The gas is ideal: PV = nRT and U = 3nRT/2.

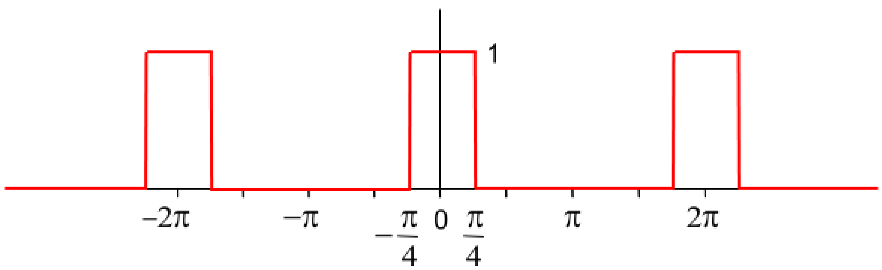
a. Let Q_{ab} represent the heat flow for the path a-b. A positive value means that heat flows into the gas; a negative indicates heat flows out. Simply state whether Q_{ab}, Q_{bc}, and Q_{ca} are negative, zero, or positive.

b. What is the efficiency of this engine if the heat that flows in during a cycle is $Q_{in} = 6nRT_0$?



[5] 5. States. There is a family of 7 people. How many ways can the following arrangement take place: 2 watching television, 3 having a snack in the kitchen, and 2 walking on a trail in the backyard.

[15] 6. Fourier Series. Find the Fourier Series for the periodic wave shown below. Your basic cycle for this repeating pattern is defined over our standard region $-\pi \le x \le \pi$, where the pulse is 1/4 the period (or wavelength) of the periodic wave.



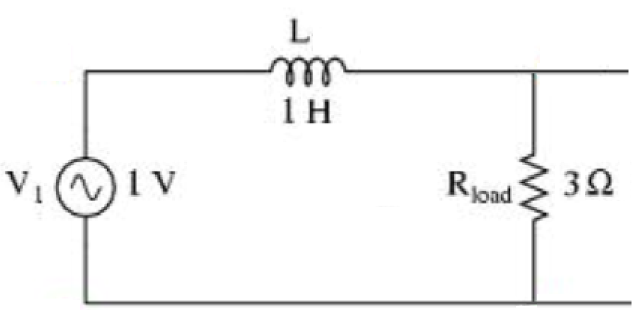
Pulse Train with a 25% Duty Cycle

Credit for this problem is heavily weighted on your explicitly writing out your answer by giving the first eight nonzero terms. You must start by writing "f(x) =" and then give the coefficients multiplied by the appropriate trig function for 8 nonzero terms, where each coefficient must be in simplest mathematical form with fractions and/or radicals.

[10] 7. Laplace Transforms. Use the real-imaginary trick to find the Laplace transforms of $\cos \omega t$ and $\sin \omega t$.

[10] 8. Convolution. Calculate $f * g$ where $f(t) = 1$ and $g(t) = t^2$.

[10]. 9. Transfer Function. A voltage $V_0 = \sin \omega t$ is applied to the LR circuit. The impedance of the inductor L is given by $Z_L = j\omega L$ and the impendence for the resistor is $Z_R = R$. Note $j = \sqrt{-1}$. Find the transfer function for this circuit. Then, find the magnitude of the output voltage when $\omega = 4$ with the values $L = 1$ and $R = 3$ given in the circuit. Is your filter low-pass or high-pass?



[15] 10. Complex Integration. Integrate $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx$ by first setting up an integral using the real-imaginary trick. Then, do the integral with complex integration techniques.

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[5] 1. Groups and Matrices. The secret here is to analyze using the closure property since we have our identity element in M_1. So we need to find M_2^2.

$$M_3 = M_2^2 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}-\frac{3}{4} & -\frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4} & -\frac{3}{4}+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

Do you recognize these as rotation matrices $R(0°)$, $R(120°)$, and $R(240°)$?

[10] 2. Relativity and Expanding.

$$\beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2} \approx (\beta_1+\beta_2)(1-\beta_1\beta_2) = (\frac{1}{10}+\frac{1}{10})(1-\frac{1}{10}\frac{1}{10}) = \frac{2}{10}\frac{99}{100} = \frac{99}{500}$$

$$\beta_{exact} = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2} = \frac{2/10}{1+1/100} = \frac{2/10}{101/100} = \frac{2}{10}\frac{100}{101} = \frac{100}{505} = \frac{20}{101}$$

[5] 3. Statistics. $< x >= \int_0^1 xP(x)dx = \int_0^1 x(2x)dx = \int_0^1 2x^2 dx = \left.\frac{2x^3}{3}\right|_0^1 = \frac{2}{3}$

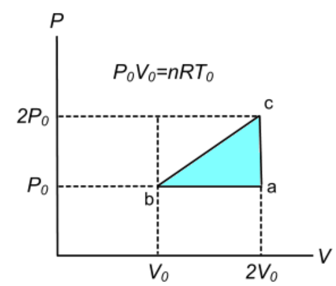
[15] 4. Engine. Given: $Q_{in} = 6nRT_0$

$Q_{ab} = U_{ab} + W_{ab} < 0$ as $U_{ab} < 0$ (T drop) and $W_{ab} < 0$

$Q_{bc} = U_{bc} + W_{bc} > 0$ as $U_{bc} > 0$ (T gain) and $W_{bc} > 0$

$Q_{ca} = U_{ca} + W_{ca} < 0$ as $U_{ca} < 0$ (T drop), $W_{ca} = 0$

Net W (blue): $W = P_0V_0/2 = nRT_0/2$ $\eta = \frac{W}{Q_{in}} = \frac{1/2}{6} = \frac{1}{12}$



[5] 5. States. $n = \frac{N!}{n_1!n_2!n_3!} = \frac{7!}{2!3!2!} = \frac{7\cdot6\cdot5\cdot4}{2\cdot1\cdot2\cdot1} = 7\cdot6\cdot5 = 7\cdot30 = 210$

[15] 6. Fourier Series. Pulse Train with a 25% Duty Cycle (symmetric). This is an even function. So this means we have the constant and cosines.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi/4} dx = \frac{2}{\pi}x\Big|_0^{\pi/4} = \frac{2}{\pi}\frac{\pi}{4} = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi/4}\cos(nx)\,dx = \frac{2}{\pi}\left.\frac{\sin(nx)}{n}\right|_0^{\pi/4} = \frac{2}{\pi}\frac{\sin(n\pi/4)}{n}$$

For n = 1, 2, 3, ... 10, $\sin(n\pi/4)$ gives $\frac{\sqrt{2}}{2}$, 1, $\frac{\sqrt{2}}{2}$, 0, $-\frac{\sqrt{2}}{2}$, -1, $-\frac{\sqrt{2}}{2}$, 0, $\frac{\sqrt{2}}{2}$, 1

$$f(x) = \frac{1}{4} + \frac{1}{\pi}(\sqrt{2}\cos x + \cos 2x + \frac{\sqrt{2}}{3}\cos 3x - \frac{\sqrt{2}}{5}\cos 5x$$
$$-\frac{1}{3}\cos 6x - \frac{\sqrt{2}}{7}\cos 7x + \frac{\sqrt{2}}{9}\cos 9x + \frac{1}{5}\cos 10x + \ldots)$$

[10] 7. Laplace Transforms.

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{i\omega t}e^{-st}dt = \left.\frac{1}{i\omega - s}\right|_0^\infty = \frac{1}{s - i\omega}$$

$$F(s) = \frac{1}{s - i\omega} = \frac{s + i\omega}{s^2 + \omega^2} \qquad L\{\cos\omega t\} = \frac{s}{s^2 + \omega^2} \qquad L\{\sin\omega t\} = \frac{\omega}{s^2 + \omega^2}$$

[10] 8. Convolution. $f * g = \int_0^t f(u)g(t-u)du = \int_0^t (t-u)^2 du = \int_0^t (t^2 - 2tu + u^2)du$

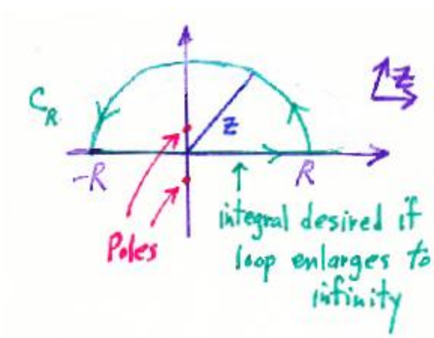
$$f * g = \left.(t^2 u - 2t\frac{u^2}{2} + \frac{u^3}{3})\right|_0^t = t^3 - t^3 + \frac{t^3}{3} = \frac{t^3}{3}$$

[10]. 9. Transfer Function. $H(\omega) = \frac{V_R}{V_L + V_R} = \frac{R}{j\omega L + R}$

$$|H(\omega)| = \left|\frac{R}{j\omega L + R}\right| = \frac{R}{\sqrt{\omega^2 L^2 + R^2}} = \frac{3}{\sqrt{4^2 \cdot 1^2 + 3^2}} = \frac{3}{\sqrt{25}} = \frac{3}{5}$$

The filter is low-pass since $\lim_{\omega\to 0}|H(\omega)| = \lim_{\omega\to 0}\frac{R}{\sqrt{\omega^2 L^2 + R^2}} = \frac{R}{\sqrt{R^2}} = 1$.

[15] 10. Complex Integration.



$$I = \int_{-\infty}^{\infty}\frac{\cos x}{x^2 + 1}dx = \mathrm{Re}\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2 + 1}dx$$

$I = \int_{-\infty}^{\infty}\frac{e^{ix}}{x^2 + 1}dx$ (the odd sine part gives 0).

$$I = \oint\frac{e^{iz}}{z^2 + 1}dz = \oint\frac{e^{iz}}{(z+i)(z-i)}dz$$

$$I = 2\pi i Res\left[\frac{e^{iz}}{(z+i)(z-i)}, i\right] = 2\pi i\left.\frac{e^{iz}}{z+i}\right|_{z=i} = 2\pi i\frac{e^{-1}}{2i} = \frac{\pi}{e}$$