Class L The Dirac Equation

L1. Review

Newton $\frac{p^2}{2m} + V = E$ Classical Mechanics

($\frac{p^2}{2m}$ ← Kinetic Energy; V ← Potential Energy; E ← Total Energy)

Schrödinger $-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi = i\hbar\frac{\partial \psi}{\partial t}$ Quantum Mechanics

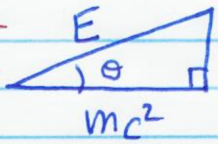
$$E \to i\hbar\frac{\partial}{\partial t} \qquad p \to -i\hbar\frac{\partial}{\partial x}$$

Pauli
Spin put in by hand $-\frac{\hbar^2}{2m}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\frac{\partial^2}{\partial x^2}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} + \begin{bmatrix}V_{11} & V_{12}\\ V_{21} & V_{22}\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} = i\hbar\frac{\partial}{\partial t}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}$

Same as $-i\hbar\nabla$

L2. Klein-Gordon Equation

Relativistic Equation



$$m^2c^4 + p^2c^2 = E^2 \qquad p \to \frac{\hbar}{i}\nabla \quad E \to i\hbar\frac{\partial}{\partial t}$$

$$m^2c^4\psi + \left(\frac{\hbar}{i}\right)^2\nabla\cdot\nabla c^2\psi = (i\hbar)^2\frac{\partial^2\psi}{\partial t^2}$$

$$m^2c^4\psi - \hbar^2c^2\nabla^2\psi = -\hbar^2\frac{\partial^2\psi}{\partial t^2}$$

Note the appearance of $\hbar$ + c
Quantum → $\hbar$; c ← Relativity

This equation doesn't describe electrons. Schrödinger tried it first.

Note the relativistic form with space + time on equal footing – both 2nd derivatives.

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{m^2c^2}{\hbar^2}\psi$$

→ Dimensions work out. And the minus sign with time. Relative minus space + time. Relativistic

$$\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \Rightarrow \left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi = 0$$

$\Box$ ← The d'Alembertian or d'Alembert operator

L3. The Dirac Equation

$$E\psi = \sqrt{p^2c^2 + m^2c^4}\,\psi$$

$$i\hbar\frac{\partial\psi}{\partial t} = c\sqrt{-\hbar^2\nabla^2 + m^2c^2}\,\psi$$

Dirac took the square root of a differential equation! → Or at least the square root of a differential operator.

$$\sqrt{p^2c^2 + m^2c^4} = c\left(\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta mc^2\right)$$

$$p^2c^2 + m^2c^4 = c^2\left(\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta mc^2\right)\left(\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta mc^2\right)$$

No way with regular algebra. Dirac goes on!

The 2×2 Pauli matrices are not enough with 2×2 matrices since we cannot find a 2×2 for β. So we go to 4 dim, 4×4.

Dirac interprets as matrices. Must anticommute.

$$\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = 1 \qquad \alpha_j\alpha_k + \alpha_k\alpha_j = 1 \quad j \neq k \qquad \{\alpha_j, \beta\} = 0$$

Pauli matrices serve as a guide.

$$\alpha_1 = \begin{bmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{bmatrix} \quad \alpha_2 = \begin{bmatrix} 0&0&0&-i \\ 0&0&i&0 \\ 0&-i&0&0 \\ i&0&0&0 \end{bmatrix} \quad \alpha_3 = \begin{bmatrix} 0&0&1&0 \\ 0&0&0&-1 \\ 1&0&0&0 \\ 0&-1&0&0 \end{bmatrix} \quad \beta = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&-1&0 \\ 0&0&0&-1 \end{bmatrix}$$

can find a β.

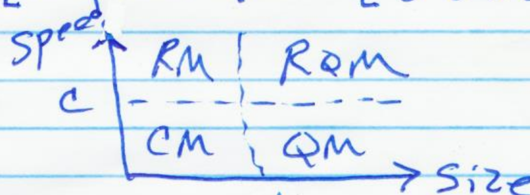
Try one $\alpha_1\alpha_2 + \alpha_2\alpha_1$

$$\begin{bmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{bmatrix}\begin{bmatrix} 0&0&0&-i \\ 0&0&i&0 \\ 0&-i&0&0 \\ i&0&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&-i \\ 0&0&i&0 \\ 0&-i&0&0 \\ i&0&0&0 \end{bmatrix}\begin{bmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{bmatrix}$$

$$\begin{bmatrix} i&0&0&0 \\ 0&-i&0&0 \\ 0&0&i&0 \\ 0&0&0&-i \end{bmatrix} + \begin{bmatrix} -i&0&0&0 \\ 0&i&0&0 \\ 0&0&-i&0 \\ 0&0&0&i \end{bmatrix} = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \equiv 0$$

L4. Spin + Antimatter

Marriage RM + QM ⟹ children



Dirac Eq. $c(\vec{\alpha}\cdot\vec{p} + \beta mc)\psi = E\psi$ For particle at rest $\vec{p} = 0$

$$\beta mc^2\psi = E\psi$$

Diagonalized, so we can read off the eigenvalues.

$$mc^2\begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&-1&0 \\ 0&0&0&-1 \end{bmatrix}\psi = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}E\psi$$

Four solutions with eigenvalues +1, +1, −1, −1

$$\psi_1 \sim \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \psi_2 \sim \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \psi_3 \sim \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \psi_4 \sim \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Electron spin up, Electron spin down — Matter

Positron up, positron down — Antimatter

Eigenvalues mc^2 for ψ_1 & ψ_2; $-mc^2$ for ψ_3 & ψ_4

Negative energies problematic. Later Quantum Field Theory.