March 26, 2020

Class R Convolution

R1. Radioactive Decay Text example is RC Circuit

$$A(t) = A_0 e^{-\lambda t}$$ Amount left (not decayed)

R2. Radioactive Waste Important Environmental Science

Dumping function $f(t)$ Added radioactive substance immediately decays on pile

$u_1 \quad u_2 \quad u_i \quad u_{i+1} \quad u \to u$ time

$\Delta u \qquad t$

$$\hookrightarrow \underline{f(u_i)\Delta u}\, e^{-\lambda(t-u_i)}$$ decay kicks in at $t = u_i$

Amount dumped in time Δu

$$A(t) = \sum_{i=1}^{n} f(u_i)\, e^{-\lambda(t-u_i)} \Delta u$$

$$A(t) = \int_0^t f(u)\, e^{-\lambda(t-u)}\, du$$

If you want, you can factor out the $e^{-\lambda t}$ from the integral.

R3. Convolution $$f(t) * g(t) \equiv \int_0^t f(u)\, g(t-u)\, du$$

Nontrivial. Note: $f(t) * 1 = \int_0^t f(u)\, du$ ← Total Dumped stays constant.

└ no radioactive decay

R4. Commutation

$$f(t) * g(t) = \int_0^t f(u)\, g(\underbrace{t-u}_{z})\, du$$

└ $t - z$ └ $-dz$

$z = t - u \qquad u = t - z$

$0 \to u \to t$

$t \to z \to 0$

$$f(t) * g(t) = \int_t^0 f(t-z)\, g(z)\, [-dz]$$

Flip integration limits ⇒ minus sign

$$= \int_0^t g(z)\, f(t-z)\, dz$$

$$= g(t) * f(t)$$

$t=u+v$ lines are $u+v=b$ constant

$v=\underbrace{-u}_{\text{slope is } -1}+b$ — b is the y-intercept (v-intercept)

R5. Convolution + Laplace Transforms

$$F(s)=\int_0^\infty f(u)e^{-su}\,du \qquad G(s)=\int_0^\infty g(v)e^{-sv}\,dv$$

$$F(s)G(s)=\int_0^\infty f(u)e^{-su}\,du\int_0^\infty g(v)e^{-sv}\,dv$$

Integration is over the 1st quadrant.

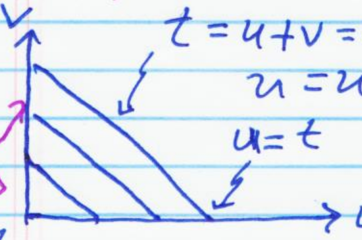
$$=\int_0^\infty\int_0^\infty e^{-s(u+v)}f(u)g(v)\,du\,dv$$

Let $t=u+v$

$du=du$

$dv=dt-du$ } $du\,dv = du\,dt$

Since $(du)^2$ can be dropped, vanishing faster.



$t=u+v=\text{const}$ } new variables

$u=u$

$u=t$ $0\to u\to t$

$0\to t\to\infty$

covers the 1st quadrant

$$F(s)G(s)=\int_{t=0}^\infty e^{-st}\underbrace{\int_0^t f(u)g(t-u)\,du}_{\text{Convolution}}\,dt$$

$$L\{f(t)\}L\{g(t)\}=L\{f(t)*g(t)\}$$

Product of Laplace Transforms = Laplace Transform of the Convolution

R6. Convolution and Power Series

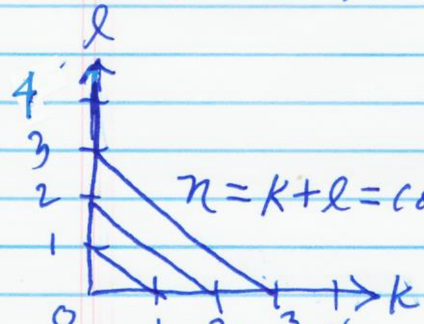
$$A(x)\,B(x)=\sum_{k=0}^\infty a_k x^k\sum_{l=0}^\infty b_l x^l$$

$$=\sum_{k=0}^\infty\sum_{l=0}^\infty a_k b_l x^{k+l}$$

$n=k+l$

$$=\sum_{n=0}^\infty\underbrace{\sum_{k=0}^n a_k b_{n-k}}_{c_n}x^n$$

$l=n-k$



$$c_n=\underbrace{\sum_{k=0}^n a_k b_{n-k}\,\Delta k}_{\int_0^n a(k)b(n-k)\,dk}\quad \Delta k = 1$$

$$c(n)=\int_0^n a(k)\,b(n-k)\,dk$$

equivalent to

$$f(t)*g(t)=\int_0^t f(u)\,g(t-u)\,du$$ Convolution