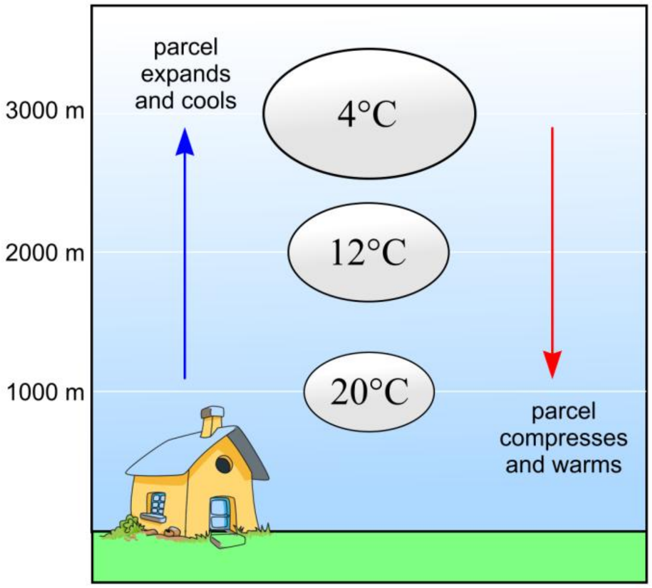
Lapse Rate. Prof. Ruiz (Doc), UNC-Asheville (1978-2021), DoctorPhys on YouTube.
Prerequisites: Chemistry, Physics I, II. Calculus I, II, III. Thermodynamics.

Chapter A. Syllabus. The goal of this course is to derive the reversible moist adiabatic lapse-rate formula and from this formula arrive at four simpler lapse-rate formulas. My courses usually require one year of calculus-based physics and three calculus courses. But this one has additional prerequisites: courses in chemistry and thermodynamics. The thermodynamics course can be any of the ones taken by physics majors, atmospheric science (meteorology) majors, or engineering majors.

I am including an extensive review of the thermodynamics required for our study (Chapters B through T). Therefore, an ambitious student without a thermodynamics course may understand the material if willing to consult a reference or two here and there when necessary. If you have a solid background in atmospheric thermodynamics, you could skip to the derivation of the reversible moist adiabatic lapse rate (Chapters U through Y). But I recommend you read through the review chapters first. Chapter Z shows how our general formula can reduce to simpler lapse-rate formulas commonly found in textbooks when suitable approximations are made.

Moist air consists of dry air, water vapor, and liquid water droplets (we do not consider ice crystals). Imagine a parcel of moist air at maximum saturation, i.e., with the most water vapor possible at a given temperature and pressure. Since we are at maximum saturation, some of the water vapor condenses to liquid water, while at the same time some of the water droplets evaporate. Our most general case will be considering very tiny water droplets that do not fall out of the parcel. They are suspended in the air. If we imagine an infinitesimal change in the parcel, the reversed infinitesimal change can occur. Such an ideal process is said to be *reversible*. In calculating lapse rate we consider our air parcel rising without exchanging heat with its surroundings – an adiabatic process.

The lapse rate Γ is defined to be negative the change in temperature with respect to increasing altitude, i.e., $\Gamma = -\frac{dT}{dz}$. The negative sign is inserted in the definition because the word “lapse” implies a decrease in temperature as the saturated parcel rises. The lapse rate for the case described above is called the *reversible saturated adiabatic lapse rate* or the *reversible moist adiabatic lapse rate*. The figure below helps us visualize this lapse rate. As an air parcel rises, the surrounding atmospheric pressure is lower due to the higher altitude and the parcel expands. The gas loses energy as it pushes outward, causing a drop in temperature. The actual surrounding atmospheric temperature also decreases with height, but this lapse rate, which is measured by weather balloons, is called the *environmental lapse rate*. Adiabatic lapse rates are relevant in the study of clouds since rising saturated air behaves adiabatically when a cloud forms.

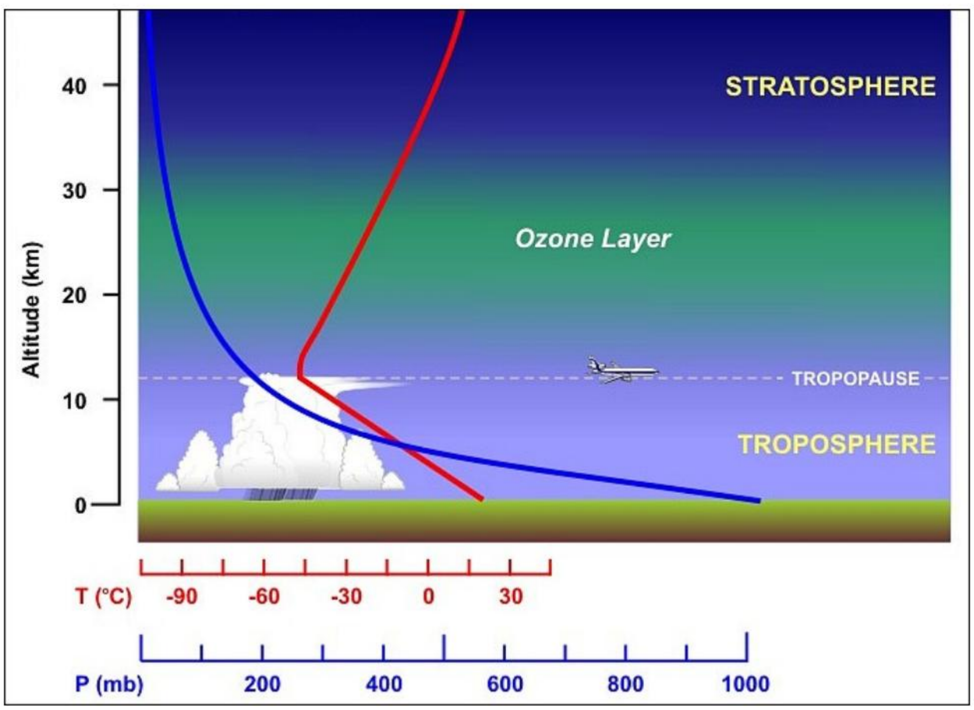


Adiabatic Rise/Descent of Air Parcel

Figure by Author, House Drawing Public Domain from rawpixel.com), House not to Scale.

Such a decrease in temperature with altitude, i.e., lapse rate, occurs in our troposphere. See the figure below for a sketch of temperature versus altitude for the lower parts of the Earth's atmosphere.

Our weather takes place mainly in the troposphere. In this layer, temperature decreases with altitude. See the red curve. The blue shows the pressure.



Credit: David Babb @ Penn State, Licensed under CC BY-NC-4.0

There are two main saturated adiabatic lapse rates found online in the American Meteorological Society (AMS) Glossary of Meteorology. These lapse rates are 1)the reversible moist-adiabatic lapse rate, which the glossary designates as Γ_{rm}, and 2)the pseudoadiabatic saturated lapse rate (Γ_{ps}), which has a popular approximate version, often just called the moist-adiabatic lapse rate (Γ_m). The subscripts are respectively "rm" = reversible moist, "ps" = pseudoadiabatic saturated, and "m" = moist.

These lapse rates are briefly described below.

1. Γ_{rm} – reversible, tiny water droplets are in suspension.

2a. Γ_{ps} – pseudoadiabatic, tiny water droplets fall out of the parcel (irreversible).

2b. Γ_m – same as 2a, but with additional approximations.

We are interested in the derivation of the first lapse rate listed above, the reversible saturated adiabatic lapse rate. The other lapse rates can be derived from this general lapse rate when suitable approximations are applied. The formula for the reversible moist adiabatic lapse rate is given below, using notation very close to that found in the Glossary of Meteorology. I explain my notation tweaks below.

$$\Gamma_{rm} = g \frac{(1 + r_v + r_l)\left(1 + \dfrac{l_v r_v}{R_d T}\right)}{c_{pd} + r_v c_{pv} + r_l c_{pl} + \dfrac{l_v^2 r_v (\varepsilon + r_v)}{R_d T^2}}$$

I will give a more detailed description of the above parameters in our course before we attack the derivation. In the above equation g is the gravitational constant of acceleration near the Earth's surface, c_{pd} is the specific heat at constant pressure for dry air, c_{pv} is the specific heat at constant pressure for water vapor, c_{pl} is the specific heat at constant pressure for liquid water, r_v is the mixing ratio of water vapor to dry air, r_l is the mixing ratio of liquid water to dry air, l_v is the enthalpy of vaporization, R_d is the gas constant for dry air, $\varepsilon = R_d / R_v = 0.622$ is the ratio of the dry-air gas constant to the water-vapor gas constant, and T is temperature in kelvins. We will cover all this in our text.

If you check the Glossary of Meteorology you will find that I have made a few slight changes in notation. The glossary gives the factor $(1 + r_t)$ in the numerator, while I give $(1 + r_v + r_l)$. The glossary groups $r_v + r_l$ together as r_t where the "t" stands for the total water content. I prefer to write out $r_v + r_l$ to make things absolutely clear. I include the subscript d for the dry air gas constant R_d and I include the subscript pl in c_{pl} for the specific heat at constant pressure for liquid water. When you see R without the subscript in the AMS Glossary of Meteorology, it is understood to represent the gas constant for dry air. The constant R is used in chemistry to stand for the ideal gas constant, for which meteorologists use R^*. Therefore, including the subscript "d" for the dry-air gas constant R_d makes these notes friendlier to students coming from chemistry and physics. It is also the convention used in practically all textbooks on the subject. I also changed the glossary's use of upper case L_v for the enthalpy of vaporization to lower case l_v since I like to be consistent using lowercase for specific quantities, i.e., per mass (kg or g).

The pressure subscript is often dropped in c_{pl} for the liquid water since this value is pretty much the same as the specific heat at constant volume for liquid water. When water absorbs heat energy, its volume stays rather constant compared to a gas absorbing heat energy and expanding. In fact, the AMS Glossary just gives c. It doesn't hurt to include the pl subscript. I have seen some authors use "*w*" for liquid water. I like to avoid "*w*" for liquid water since "*w*" may suggest to some students all the phases for H_2O. Also, some authors use "*w*" for the water vapor mixing ratio. We will use "r_v" following the AMS Glossary.

Some editors point out that when subscripts are themselves variables such as pressure, they should be in italics. When subscripts refer to descriptions like "dry" or "liquid" they should not be in italics. I will not follow this strict convention for three reasons: 1)it takes too long to format formulas in this way, 2)the non-italics for "l" looks like a "1" with my MathType default equation font, and 3)the reputable AMS as well as many authors of textbooks use italics for the subscripts regardless. If I have a subscript that is a word, I might decide to break the rule if the word looks weird in italics.

After we derive the reversible saturated adiabatic lapse rate, we will show that with appropriate approximations this general lapse rate reduces to four other lapse rates. All of these will be explained in detail in our course. To work through all this material we need to agree on notation. Since authors will differ on notation, I carefully lay out my notation below.

There is no convention on abbreviations that everyone uses. I will employ the following abbreviations.

RSALR - Reversible Saturated Adiabatic Lapse Rate (vapor droplets suspended in air)

PSALR - Pseudoadiabatic Saturated Lapse Rate (vapor droplets fall out, we ignore them)

MSALR - Moist Saturated Adiabatic Lapse Rate (PSALR simplified with approximations)

MUSALR - Moist Unsaturated Adiabatic Lapse Rate

DALR - Dry Adiabatic Lapse Rate

The follow diagram gives a map of how the most general lapse-rate formula reduces to other forms when approximations are made.

$$\Gamma_{rm} = g\frac{\left(1+r_v+r_l\right)\left(1+\dfrac{l_v r_v}{R_d T}\right)}{c_{pd}+r_v c_{pv}+r_l c_{pl}+\dfrac{l_v^2 r_v\left(\varepsilon+r_v\right)}{R_d T^2}}$$

Reversible Saturated Adiabatic Lapse Rate

RSALR

$$\Gamma_{ps} = g\frac{\left(1+r_v\right)\left(1+\dfrac{l_v r_v}{R_d T}\right)}{c_{pd}+r_v c_{pv}+\dfrac{l_v^2 r_v\left(\varepsilon+r_v\right)}{R_d T^2}}$$

Pseudoadiabatic Saturated Lapse Rate

PSALR

$$\Gamma_m = \frac{g}{c_{pd}}\frac{1+\dfrac{l_v r_v}{R_d T}}{1+\dfrac{\varepsilon l_v^2 r_v}{c_{pd} R_d T^2}}$$

Moist Saturated Adiabatic Lapse Rate

MSALR

Moist Unsaturated Adiabatic Lapse Rate

$$\Gamma_{m,unsat} = g\frac{1+r_v}{c_{pd}+r_v c_{pv}}$$

MUALR

Dry Adiabatic Lapse Rate

$$\Gamma_d = \frac{g}{c_{pd}}$$

DALR

Notation (I will discuss these parameters in our chapters.)

pressure, temperature (kelvins), volume	p , T , V
mass, volume, specific volume	m , V , $\alpha = V/m$
density	$\rho = m/V$
altitude (geometric)	z
ideal gas law	$pV = nR^*T$
Avogadro's number	N_A
energy, specific energy	U , $u = U/m$
enthalpy, specific enthalpy	$H = U + pV$, $h = u + p\alpha$
enthalpy of evaporation	l_v
entropy and specific entropy	S and $s = S/m$
ideal gas constant in $pV = nR^*T$	R^*
gas constant	$R = R^*/M$
gas constant for dry air, water vapor	$R_d = R^*/M_d$, $R_v = R^*/M_v$
gas constant for moist air	$R_m = R^*/M_m$
gas constant ratio	$\varepsilon = R_d / R_v$
Gibbs free energy, specific Gibbs energy	$G = U - TS + pV$, $g = u - Ts + p\alpha$
heat transfer, specific heat transfer	δQ , $\delta q = \delta Q / m$
ideal gas constant	R^*
lapse rate	$\Gamma = -dT/dz$
mixing ratio (vapor): vapor mass/dry air mass	$r_v = m_v / m_d$
mixing ratio (liquid water)	$r_l = m_l / m_d$
molecular weight (molar mass)	M
molecular weight of dry air, water vapor	M_d , M_v
molecular weight of moist air	M_m
number of moles	n
partial pressure of dry air	p_d
partial pressure of water vapor	e
pressure of water vapor at saturation	e_s
specific heat at constant: volume, pressure	c_V , c_p
specific heat of dry air at constant p	c_{pd}
specific heat of vapor at constant p	c_{pv}
specific heat of liquid H_2O at constant p	c_{pl}
specific humidity (do not confuse with heat q)	$q = m_v / (m_d + m_v)$
virtual temperature, density temperature	T_v , T_ρ
work done by gas	$\delta W = pdV$, $\delta w = pd\alpha$

Prerequisite Formulas. These notes build on a background in chemistry (one semester), physics (two semesters), some thermodynamics (preferably one semester), mathematics (three semesters of calculus, a preferred fourth semester in differential equations), and a course in thermodynamics. Below are the formulas we will need in our derivation of the reversible saturated adiabatic lapse rate. I will provide a detailed review of these formulas in Chapters B through U. When I use them in our derivation (Chapters V through Y) I will refer to them by number. I recommend printing this page out and having it as a handy reference.

(1) $\boxed{pV = nR^*T}$ (2) $\boxed{p\alpha = RT}$ (3) $\boxed{R = \frac{R^*}{M}}$ (4) $\boxed{R_d = \frac{R^*}{M_d}}$ (5) $\boxed{R_v = \frac{R^*}{M_v}}$

(6) $\boxed{\varepsilon = \frac{R_d}{R_v} = \frac{M_v}{M_d}}$ (7) $\boxed{p = p_d + e}$ (8) $\boxed{p_d = \rho_d R_d T}$ (9) $\boxed{e = \rho_v R_v T}$

(10) $\boxed{\Gamma = -\frac{dT}{dz}}$ (11) $\boxed{\frac{dp}{dz} = -\rho g}$

(12) $\boxed{\delta q = du + p d\alpha}$ (13) $\boxed{h = u + p\alpha}$ (14) $\boxed{\delta q = dh - \alpha dp}$

(15) $\boxed{c_V = \frac{\delta q_V}{dT} = \frac{du}{dT}}$ (16) $\boxed{c_p = \frac{\delta q_p}{dT} = \frac{dh}{dT}}$

(17) $\boxed{ds = \frac{\delta q}{T}}$ (18) $\boxed{l_v = h_v - h_l}$ (19) $\boxed{\frac{dl_v}{dT} = c_{pv} - c_{pl}}$

(20) $\boxed{g = u - sT + p\alpha}$ (21) $\boxed{dg = -sdT + \alpha dp}$ (22) $\boxed{\left(\frac{\partial s}{\partial p}\right)_T = -\left(\frac{\partial \alpha}{\partial T}\right)_p}$

(23) $\boxed{\frac{dp}{dT} = \frac{l_v}{T\Delta\alpha}}$ (24) $\boxed{\frac{de_s}{dT} = \frac{e_s l_v}{R_v T^2}}$ (25) $\boxed{r_v = \frac{\varepsilon e}{p_d}}$ (26) $\boxed{r_v = \frac{\varepsilon e}{p - e}}$

(27) $\boxed{T_v = \frac{1 + \frac{r_v}{\varepsilon}}{1 + r_v} T}$ (28) $\boxed{T_\rho = \frac{1 + \frac{r_v}{\varepsilon}}{1 + r_v + r_l} T}$

(29) $\boxed{ds_d = \frac{c_{pd}}{T} dT - \frac{R_d}{p_d} dp_d}$ (30) $\boxed{ds_v = \frac{c_{pv}}{T} dT - \frac{R_v}{e_s} de_s}$ (31) $\boxed{ds_l = \frac{c_{pl}}{T} dT}$

(32) $\boxed{S = m_d s_d + m_v s_v + m_l s_l}$