Lapse Rate. Prof. Ruiz (Doc), UNC-Asheville (1978-2021), DoctorPhys on YouTube.
Prerequisites: Chemistry, Physics I, II. Calculus I, II, III. Thermodynamics.

Chapter B. Ideal Gas. You can think of the ideal gas law as incorporating three laws. We consider a fixed amount of gas in an enclosure. The gas has a volume (V), temperature (T), and pressure (p). We hold each one of these constant in order to state the three laws. But for these laws, we must use the Kelvin temperature scale, the scale where zero means absolute zero, the coldest temperature possible. This temperature 0 K is equivalent to -273.15 °C and -459.67 °F. In the old days they thought all motion stopped at this temperature. However, due to quantum mechanics (1925), we now say the motion is minimal or the system is in its lowest energy state.

1. The Ideal Gas Law.

a. Boyle's Law. At constant temperature, the pressure of a gas is inversely related to its volume.

$$p \sim \frac{1}{V}$$

We can express this relation as

$$(pV)_T = const.$$

The subscript "T" reminds us that the temperature is held constant.

b. Charles's Law. At constant pressure, the volume of a gas is proportional to its temperature.

$$V \sim T$$

Think of a parcel of gas that can expand or contract. If you heat the gas gently, the gas will expand if the same pressure is to be maintained. We can express this relation as

$$\left(\frac{V}{T}\right)_p = const.$$

The subscript "p" reminds us that the pressure is held constant.

c. Gay-Lussac's Law. At constant volume, the pressure of a gas is proportional to its temperature.

$$p \sim T$$

We can express this relation as

$$\left(\frac{p}{T}\right)_V = const.$$

The subscript "V" reminds us that the volume is held constant.

Think of the gas now in a rigid container so that the gas cannot change volume. If we heat the gas, the pressure will increase.

We can incorporate all these laws in the form: $\frac{pV}{T} = const$

d. Avogadro's Law. At this point I would like to add a fourth law – Avogadro's Law, which states that under the same pressure, volume, and temperature, all gases contain the same number of molecules. If you want to double the volume and have the same pressure and temperature, we can achieve this result by combining two equal volumes of the gas with the same pressure and temperature. We will then have double the number of gas particles.

We can then include the number of particles N as follows:

$$\frac{pV}{T} = kN,$$

where N is the number of particles and k is a constant called the Boltzmann constant.

Since there are so many gas particles in a gas, chemists use moles, designated by n, instead of particles. A mole consists of Avogadro's number of entities. Avogadro's number is exactly equal to $6.022\ 140\ 76 \times 10^{23}$. This exact definition has been in existence since 2019. Before that date, the number was defined as the number of carbon atoms in 12 grams of carbon-12, which is very close to the above exact number. The 2019 definition makes the number precise and independent of the reference to carbon-12. *Avogadro's constant* is defined with this number and the unit inverse mol (mole) attached to it.

$$N_A = 6.022\ 140\ 76 \times 10^{23}\ \frac{1}{\text{mol}}.$$

Note that Avogadro's number, i.e., $6.022\ 140\ 76 \times 10^{23}$ has no units. It is just a number.

But, Avogadro's constant includes the unit $\frac{1}{\text{mol}}$.

When you use the number of moles n, we have a different constant k' in the ideal gas formula.

$$\frac{pV}{T} = k'n$$

The ideal gas law is often written as

$$pV = nk'T.$$

The constant k' is called the ideal gas constant and here is where chemists and meteorologists use different symbols. Chemists use R and meteorologists use R^*. Why? Meteorologists reserve R for something more convenient to them., which you will see shortly. The ideal gas law is then

$$pV = nR^*T .$$

I will label this equation as (1) for future reference.

(1) $\boxed{pV = nR^*T}$

2. The Ideal Gas Law for Meteorologists. We will now cast the ideal gas law in the form that meteorologists use. Meteorologists prefer to work with volumes per mass as the atmosphere is too vast to work with any kind of total volume. Start with some volume V for our gas. Then divide the ideal gas equation by the mass m of the gas in our volume V .

$$pV = nR^*T \quad \Rightarrow \quad p\frac{V}{m} = \frac{n}{m}R^*T$$

In physics, $\frac{V}{m}$ is often written as a lowercase v. This volume is an *intensive variable* or *intensive* property as it is the same no matter how much of our gas we are considering. In contrast, the total volume is called an *extensive variable* or *extensive property*. The atmospheric scientist or meteorologist label the intensive volume with the symbol α , Therefore,

$$\alpha = \frac{V}{m} .$$

The lowercase v will often appear as a subscript for vapor. However, there will be times when the subscript v will mean constant volume. So, we will need to be careful.

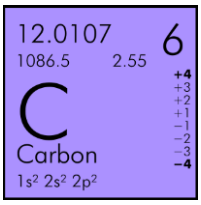
Note that by nature, the pressure p and temperature T do not depend on which percent of a uniform gas we focus on. They are intensive variables. These variables have the same value everywhere in a gas that has settled down and is in equilibrium. Now, we have three intensive variables: p , α , and T .

With $\alpha = \frac{V}{m}$, we will transform the ideal gas law. Divide the ideal gas law by m .

$$pV = nR^*T \quad \Rightarrow \quad p\frac{V}{m} = \frac{n}{m}R^*T \quad \Rightarrow \quad p\alpha = \frac{n}{m}R^*T$$

Let's focus on $\frac{n}{m}R^*$. If you take the mass m and divide by the number of moles n you get the mass per mole, which we designate by M.

$$M = \frac{m}{n}$$



This M is also called the *molar mass*. In the Periodic Table values are listed for each element where an average value is taken over the isotopes. For example, the common form of carbon is carbon-12. To be a carbon atom, the nucleus must have 6 protons. The 12 means there are 6 neutrons since the 12 indicates the sum of protons and neutrons in the nucleus. Carbon can also have 8 neutrons in the nucleus. In that case, we have carbon-14. A weighted average is taken over isotopes where the weighting factor depends on the percentage of each isotope occurring in nature. The molar mass M value taken from the Periodic Table is also called the *atomic weight*. See the element carbon at the left from the periodic table. The atomic weight is 12.0107, which is very close to 12 since carbon-12 is most abundant in nature.

Remember from our discussion earlier, the 12.01 means if you have 12.01 grams of carbon, you have essentially one mole, which is Avogadro's number of carbon atoms.

When we consider molecules we combine the atoms to find the molecular weight. Then, for air, we take a weighted average of the constituents that make up air.

Using $M = \frac{m}{n}$ in $p\alpha = \frac{n}{m}R^*T$ we find the following.

$$p\alpha = \frac{n}{m}R^*T \quad \Rightarrow \quad p\alpha = \frac{R^*}{m/n}T \quad \Rightarrow \quad p\alpha = \frac{R^*}{M}T\,.$$

Meteorologist define the ideal gas constant for a specific gas as $R = \frac{R^*}{M}$. The gas constant R depends on the molar mass of the gas. We can now write the ideal gas law in atmospheric science.

$$p\alpha = \frac{R^*}{M}T \quad \Rightarrow \quad p\alpha = RT$$

We have our second formula.

(2) $\boxed{p\alpha = RT}$

3. The Ideal Gas Constant for Meteorologists. I would like to emphasize that the ideal gas constant depends on the gas. Therefore, we have a third equation.

$$(3)\quad \boxed{R = \frac{R^*}{M}}$$

The gas constant depends on the molar mass M of the gas.

4. Ideal Gas Constant for Dry Air. We divide the ideal gas constant R^* by the molar mass M_d for dry air.

$$(4)\quad \boxed{R_d = \frac{R^*}{M_d}}$$

5. Ideal Gas Constant for Vapor. We divide the ideal gas constant R^* by the molar mass M_v of water vapor.

$$(5)\quad \boxed{R_v = \frac{R^*}{M_v}}$$

We will rarely introduce specific numbers in this course since the main goal is to derive the reversible saturated adiabatic lapse rate formula, which consists of symbols. Therefore, our course is heavily theoretical. We concentrate on theory and formulas. Numbers are very important in another course of mine: Atmospheric Thermodynamics.

6. Ratio of Dry Air and Water Vapor Gas Constants. The ratio of the dry-air to water-vapor gas constants arises often in atmospheric science. Therefore, a special parameter is introduced.

$$\varepsilon = \frac{R_d}{R_v}$$

I would like to express this equation in terms of the molar masses of dry air and water vapor. We apply the equation $R = \frac{R^*}{M}$ to dry air and water vapor.

$$R_d = \frac{R^*}{M_d} \qquad R_v = \frac{R^*}{M_v}$$

Then $\varepsilon = \frac{R_d}{R_v}$ becomes

$$\varepsilon = \frac{R_d}{R_v} = \frac{\frac{R^*}{M_d}}{\frac{R^*}{M_v}} = \frac{R^*}{M_d}\frac{M_v}{R^*} = \frac{M_v}{M_d}.$$

We now have our 6th equation.

$$(6)\quad \boxed{\varepsilon = \frac{R_d}{R_v} = \frac{M_v}{M_d}}$$

7. Variables. For later reference, let me talk about variables a little here. The variables pressure, volume, and temperature are *macroscopic variables* since they describe the gas system as a whole. In contrast, *microscopic variables* describe details of the gas particles. Examples of microscopic variables are masses and speeds of individual gas particles. The variables pressure, volume, and temperature are variables that describe the state of the gas overall and are directly measured in the lab. We call these *state variables*.

I would like to point out some mathematical properties of state variables. A variable such as volume (V) has a differential form dV and we can integrate over the volume from an initial volume V_1 to a final volume V_2:

$$\int_{V_1}^{V_2} dV = V\Big|_{V_1}^{V_2} = V_2 - V_1 .$$

The path in going from V_1 to V_2 does not matter. Imagine a process when the volume increases, then decreases, remains constant for awhile, then increases to its final destination. The change in volume $\Delta V = V_2 - V_1$ depends only on the end points V_1 and V_2. We call dV an *exact differential*. In our process in going from V_1 to V_2 the change is independent of the path.

You might ask why are we going through all this. Isn't this obvious? The reason I emphasize these points that you already know is that when we get to heat and work, we will find that the heat and work do not operate this way. Heat and work are path dependent and each is called an *inexact differential*.