Lapse Rate. Prof. Ruiz (Doc), UNC-Asheville (1978-2021), DoctorPhys on YouTube.
Prerequisites: Chemistry, Physics I, II. Calculus I, II, III. Thermodynamics.

Chapter H. Specific Heat at Constant Pressure. We showed in the previous chapter that

$$C = \frac{\delta Q}{dT} = \frac{dU}{dT} + p\frac{dV}{dT}.$$

For constant pressure, we have

$$C_p = \left.\frac{\delta Q}{dT}\right|_p = \left.\frac{dU}{dT}\right|_p + p\left.\frac{dV}{dT}\right|_p.$$

We already know that $U = \frac{3}{2}nR^*T$ and $\frac{dU}{dT} = \frac{3}{2}nR^*$.

But since $U = U(T)$ we have $\left.\frac{dU}{dT}\right|_p = \left.\frac{dU}{dT}\right|_V = \frac{dU}{dT}$.

We found that the derivative $\frac{dU}{dT}$ is also C_V. Therefore,

$$C_p = \left.\frac{\delta Q}{dT}\right|_p = \frac{dU}{dT} + p\left.\frac{dV}{dT}\right|_p \quad \text{can be written as} \quad C_p = \left.\frac{\delta Q}{dT}\right|_p = C_V + p\left.\frac{dV}{dT}\right|_p$$

Summary

$$\boxed{C_p = C_v + p\left.\frac{dV}{dT}\right|_p}$$

For an ideal gas $pV = nR^*T$.

Take the derivative of the ideal gas equation with respect to T in order to find $p\frac{dV}{dT}$.

$$\frac{d}{dT}(pV) = \frac{d}{dT}(nR^*T) \quad => \quad p\frac{dV}{dT} + V\frac{dp}{dT} = nR^* \quad => \quad p\frac{dV}{dT} = nR^* - V\frac{dp}{dT}$$

Since the pressure is constant, $\frac{dp}{dT} = 0$ and we have the following.

$$p\left.\frac{dV}{dT}\right|_p = nR^* - V\left.\frac{dp}{dT}\right|_p \quad => \quad p\left.\frac{dV}{dT}\right|_p = nR^* - V\cdot 0 \quad => \quad p\left.\frac{dV}{dT}\right|_p = nR^*$$

Substituting $p\left.\frac{dV}{dT}\right|_p = nR^*$ in $C_p = C_v + p\left.\frac{dV}{dT}\right|_p$.

$C_p = C_v + p\left.\frac{dV}{dT}\right|_p$ becomes $C_p = C_V + nR^*$.

Summary

$$\boxed{C_p = C_V + nR^*}$$

We now remember our formula $mR = nR^*$ and use it to obtain a pair of expressions.

$$C_p = C_V + nR^* \qquad C_p = C_V + mR$$

We arrive at the specific heat formulas for the chemist and meteorologist.

Chemists	Meteorologists
$c_{p,n} = \frac{C_p}{n} = \frac{C_V}{n} + \frac{nR^*}{n}$	$c_p = \frac{C_p}{m} = \frac{C_V}{m} + \frac{mR}{m}$
$c_{p,n} = \frac{C_V}{n} + \frac{nR^*}{n}$	$c_p = \frac{C_V}{m} + \frac{mR}{m}$
$c_{p,n} = c_{V,n} + R^*$	$c_p = c_V + R$

The first part $c_p = \frac{\delta q_p}{dT}$ is a definition. The pressure is held constant as a substance absorbs heat energy. For a great rise in temperature, the heat capacity is small; for little rise in temperature, the heat capacity is large. We use the first law of thermodynamics in the form $\delta q = dh - \alpha dp$. This substitution gives us $c_p = \frac{\delta q_p}{dT} = \left(\frac{dh}{dT}\right)_p - \alpha\left(\frac{dp}{dT}\right)_p$. Since pressure is held constant, $\left(\frac{dp}{dT}\right)_p = 0$ and $c_p = \left(\frac{dh}{dT}\right)_p$. We can drop the pressure subscript on the derivative. Here is why. The specific enthalpy is $h = u + pv$. We have stated that the energy for an ideal gas is only a function of temperature, i.e., $u(T) = c_V T$. We can express the enthalpy as $h = u(T) + p\alpha = c_V T + p\alpha$.

From the ideal gas law, $p\alpha = RT$, likewise only a function of temperature. Therefore we have for the enthalpy of an ideal gas: $h(T) = c_V T + RT$. So, we can drop the pressure subscript for our derivative and write $c_p = \left(\frac{dh}{dT}\right)_p = \frac{dh}{dT}$.

Using $c_p = \left(\frac{dh}{dT}\right)_p = \frac{dh}{dT}$ and $h(T) = c_V T + RT$,

we find

$$c_p = \frac{dh}{dT} = \frac{d}{dT}\left(c_V T + RT\right) = c_V + R .$$

I consider this equation a bonus relation, which we already derived another way above. Remember that $R = R^*/M$ if you are joining us from chemistry or physics. The specific heat for constant pressure is greater than the specific heat for constant volume since we are allowing the substance to expand (doing work) in order to keep the pressure constant.

We have our next equation.

$$\{16\} \quad \boxed{c_p = \frac{\delta q_p}{dT} = \frac{dh}{dT}}$$

Bonus Relation: $\boxed{c_p = c_V + R}$.

Compare (16) with (15).

$$(15) \quad \boxed{c_V = \frac{\delta q_V}{dT} = \frac{du}{dT}} \qquad \{16\} \quad \boxed{c_p = \frac{\delta q_p}{dT} = \frac{dh}{dT}}$$

Also, compare the following.

$$\boxed{c_p = c_V + R} \qquad \boxed{c_{p,n} = c_{V,n} + R^*}$$