Lapse Rate. Prof. Ruiz (Doc), UNC-Asheville (1978-2021), DoctorPhys on YouTube.
Prerequisites: Chemistry, Physics I, II. Calculus I, II, III. Thermodynamics.

Chapter P. Virtual Temperature. We start with the following equations we have discussed.

$$1)\ \boxed{pV = nR^*T} \quad (2)\ \boxed{p\alpha = RT} \quad (3)\ \boxed{R = \frac{R^*}{M}}$$

Since the density $\rho = \dfrac{m}{V}$ is the reciprocal of the specific volume $\alpha = \dfrac{V}{m}$, we have $\alpha = \dfrac{1}{\rho}$. We substitute this result into (2) $p\alpha = RT$.

$$p\alpha = RT \quad => \quad p\frac{1}{\rho} = RT \quad => \quad p = \rho RT$$

We now consider moist air. To remind ourselves, we put a subscript "m" for moist on our gas constant.

$$R \to R_m$$

The ideal gas law for moist air can then be written as

$$p = \rho R_m T \, .$$

To summarize: p is the total pressure, ρ is the density of the moist air, R_m is the gas constant for the moist air, and T is the temperature.

The virtual temperature is defined as the temperature that we would need for dry air to have the same total pressure and density. This definition gives us the following expression involving the virtual temperature T_v:

$$p = \rho R_m T \equiv \rho R_d T_v \, .$$

The above equation immediately gives us

$$R_m T = R_d T_v \, .$$

We note that the mass of the moist air consists of the mass of the dry air plus the mass of the vapor within the same volume V at temperature T. Therefore, we have the following mass and density relations:

$$m = m_d + m_v \quad \text{and} \quad \rho = \frac{m}{V} = \frac{m_d + m_v}{V} = \frac{m_d}{V} + \frac{m_v}{V} = \rho_d + \rho_v \, .$$

Using (7) $\boxed{p = p_d + e}$ along with the ideal gas equations (8) $\boxed{p_d = \rho_d R_d T}$ and (9) $\boxed{e = \rho_v R_v T}$ for the partial pressures, we can perform the following calculations.

$$p = p_d + e \quad \Rightarrow \quad \rho R_m T = \rho_d R_d T + \rho_v R_v T \quad \Rightarrow \quad \rho R_m = \rho_d R_d + \rho_v R_v$$

$$\Rightarrow \quad R_m = \frac{\rho_d R_d + \rho_v R_v}{\rho} \quad \Rightarrow \quad R_m = \frac{(m_d / V) R_d + (m_v / V) R_v}{(m / V)} \quad \Rightarrow \quad R_m = \frac{m_d R_d + m_v R_v}{m}$$

$$\Rightarrow \quad R_m = \frac{m_d R_d + m_v R_v}{m_d + m_v} \quad \Rightarrow \quad R_m = R_d \frac{m_d + m_v \frac{R_v}{R_d}}{m_d + m_v}$$

Remember the definition $\varepsilon \equiv \frac{R_d}{R_v}$ (3). We will use it below.

$$R_m = R_d \frac{m_d + m_v \frac{R_v}{R_d}}{m_d + m_v} \quad \Rightarrow \quad R_m = R_d \frac{\frac{m_d}{m_d} + \frac{m_v}{m_d} \frac{R_v}{R_d}}{\frac{m_d}{m_d} + \frac{m_v}{m_d}} \quad \Rightarrow \quad R_m = R_d \frac{\frac{m_d}{m_d} + \frac{m_v}{m_d} \frac{1}{\varepsilon}}{\frac{m_d}{m_d} + \frac{m_v}{m_d}}$$

Now recall the definition of the mixing ratio: $r_v = \frac{m_v}{m_d}$. We use this definition next.

$$R_m = R_d \frac{\frac{m_d}{m_d} + \frac{m_v}{m_d} \frac{1}{\varepsilon}}{\frac{m_d}{m_d} + \frac{m_v}{m_d}} \quad \Rightarrow \quad \Rightarrow \quad R_m = R_d \frac{1 + r_v \frac{1}{\varepsilon}}{1 + r_v} \quad \Rightarrow \quad \frac{R_m}{R_d} = \frac{1 + r_v \frac{1}{\varepsilon}}{1 + r_v}.$$

The virtual temperature is found from our above relation $R_m T = R_d T_v$.

$$T_v = \frac{R_m}{R_d} T \quad \Rightarrow \quad T_v = \frac{R_m}{R_d} T = \frac{1 + \frac{r_v}{\varepsilon}}{1 + r_v} T \quad \Rightarrow \quad \boxed{T_v = \frac{1 + \frac{r_v}{\varepsilon}}{1 + r_v} T}$$

(27) $\boxed{T_v = \frac{1 + \frac{r_v}{\varepsilon}}{1 + r_v} T}$ The virtual temperature for moist air.