Lapse Rate. Prof. Ruiz (Doc), UNC-Asheville (1978-2021), DoctorPhys on YouTube.
Prerequisites: Chemistry, Physics I, II. Calculus I, II, III. Thermodynamics.

Chapter Q. Density Temperature. We now consider dry air, water vapor, and suspended water droplets. The density temperature is the temperature that dry air would have to be in order to achieve the same pressure and temperature as the mixture of dry air, water vapor, and liquid water droplets. We need to be very careful here since the liquid water droplets are not a gas. We do not have an equation of state for the water. I will take the careful approach given in the text Emanuel, Kerry A. (1994). *Atmospheric Convection*. New York: Oxford University Press.

Emanuel includes water droplets and tiny ice crystals. We will not need the ice crystals. We start with the specific volume of the moist air (gas) containing suspended water droplets.

$$\alpha = \frac{V_{gas} + V_{liquid}}{m_{gas} + m_{liquid}} \equiv \frac{V_g + V_l}{m_g + m_l}$$

The mass m_g of the gas consists of the mass of the dry air m_d plus the mass m_v of the water vapor.

$$\alpha = \frac{V_g + V_l}{m_g + m_l} \quad => \quad \alpha = \frac{V_g + V_l}{(m_d + m_v) + m_l}$$

Note that the volume V_g for the gas is the same for the dry air and vapor.

We now divide top and bottom of this ratio by m_d. We find

$$\alpha = \frac{\frac{V_g}{m_d} + \frac{V_l}{m_d}}{\frac{m_d}{m_d} + \frac{m_v}{m_d} + \frac{m_l}{m_d}} \quad \text{leading to} \quad \alpha = \frac{\alpha_d + \frac{V_l}{m_d}}{1 + r_v + r_l},$$

where $\alpha_d = \frac{V_g}{m_d}$ is the specific volume of the dry air, $r_v = \frac{m_v}{m_d}$ is the vapor mixing ratio, and $r_l = \frac{m_l}{m_d}$ is the liquid mixing ratio. The specific volume of the liquid water, by definition, is $\alpha_l = \frac{V_l}{m_l}$. With these variables,

$$\alpha = \frac{\alpha_d + \frac{V_l}{m_d}}{1 + r_v + r_l} \quad => \quad \alpha = \frac{\alpha_d + \frac{m_l}{m_l}\frac{V_l}{m_d}}{1 + r_v + r_l} \quad => \quad \alpha = \frac{\alpha_d + \frac{m_l}{m_d}\frac{V_l}{m_l}}{1 + r_v + r_l} \quad => \quad \alpha = \frac{\alpha_d + r_l\alpha_l}{1 + r_v + r_l}$$

We now factor out α_d.

$$\alpha = \frac{\alpha_d + r_l \alpha_l}{1 + r_v + r_l} \quad => \quad \alpha = \alpha_d \frac{1 + r_l \dfrac{\alpha_l}{\alpha_d}}{1 + r_v + r_l}$$

Remember, that density ρ is the reciprocal of specific volume $\rho = \dfrac{1}{\alpha}$ in general. Therefore,

$$\alpha = \alpha_d \frac{1 + r_l \dfrac{\alpha_l}{\alpha_d}}{1 + r_v + r_l} \quad => \quad \alpha = \alpha_d \frac{1 + r_l \dfrac{\rho_d}{\rho_l}}{1 + r_v + r_l}.$$

The density of dry air $\rho_d = 1.225 \ \dfrac{\text{kg}}{\text{m}^3}$ and the density of liquid water is $\rho_l = 1000 \ \dfrac{\text{kg}}{\text{m}^3}$.

The ratio $\dfrac{\rho_d}{\rho_l} = \dfrac{1.225 \ \dfrac{\text{kg}}{\text{m}^3}}{1000 \ \dfrac{\text{kg}}{\text{m}^3}} \approx 10^{-3}$.

The mixing ratio for water vapor is typically $r_v = 20 \ \dfrac{\text{g}}{\text{kg}}$ with a usual range from $0 \ \dfrac{\text{g}}{\text{kg}}$ to $40 \ \dfrac{\text{g}}{\text{kg}}$. Let's say that we start with $r_v = 20 \ \dfrac{\text{g}}{\text{kg}}$ and that half of the vapor condenses to liquid water. Then $r_v = r_l = 10 \ \dfrac{\text{g}}{\text{kg}}$.

The term $r_l \dfrac{\rho_d}{\rho_l}$ in $\alpha = \alpha_d \dfrac{1 + r_l \dfrac{\rho_d}{\rho_l}}{1 + r_v + r_l}$ is therefore

$$r_l \frac{\rho_d}{\rho_l} = 10 \ \frac{\text{g}}{\text{kg}} \cdot 10^{-3} = 0.010 \frac{\text{kg}}{\text{kg}} \cdot 10^{-3} = 10^{-2} \cdot 10^{-3} = 10^{-5}.$$

The combination $1 + r_l \dfrac{\rho_d}{\rho_l}$ in $\alpha = \alpha_d \dfrac{1 + r_l \dfrac{\rho_d}{\rho_l}}{1 + r_v + r_l}$ is $1 + r_l \dfrac{\rho_d}{\rho_l} = 1 + 10^{-5}$.

We can clearly throw out the 10^{-5} term, giving us

$$\alpha = \alpha_d \frac{1}{1 + r_v + r_l}.$$

We are safe in applying the ideal gas equation to the α_d factor (dry air).

Using the ideal gas law $p_d \alpha_d = R_d T$ by substituting $\alpha_d = \frac{R_d T}{p_d}$,

$$\alpha = \alpha_d \frac{1}{1 + r_v + r_l} \quad \text{becomes} \quad \alpha = \frac{R_d T}{p_d} \frac{1}{1 + r_v + r_l}.$$

We are almost there. Now, multiply by $\frac{p}{p}$, which is 1. We can always do that.

$$\alpha = \frac{R_d T}{p_d} \frac{1}{1 + r_v + r_l} \quad => \quad \alpha = \frac{R_d T}{p_d} \frac{p}{p} \frac{1}{1 + r_v + r_l} \quad => \quad \alpha = \frac{R_d T}{p_d} \frac{p_d + e}{p} \frac{1}{1 + r_v + r_l}$$

$$\alpha = \frac{R_d T}{p} \frac{p_d + e}{p_d} \frac{1}{1 + r_v + r_l} \quad => \quad \alpha = \frac{R_d T}{p} \left(1 + \frac{e}{p_d}\right) \frac{1}{1 + r_v + r_l}.$$

At this point we use (25) $\boxed{r_v = \frac{\varepsilon e}{p_d}}$ in the form $\frac{e}{p_d} = \frac{r_v}{\varepsilon}$ so that

$$\alpha = \frac{R_d T}{p} \left(1 + \frac{e}{p_d}\right) \frac{1}{1 + r_v + r_l} \quad \text{can be written as} \quad \alpha = \frac{R_d T}{p} \left(1 + \frac{r_v}{\varepsilon}\right) \frac{1}{1 + r_v + r_l}.$$

Multiplying both sides of this equation by p,

$$p\alpha = R_d T \left(1 + \frac{r_v}{\varepsilon}\right) \frac{1}{1 + r_v + r_l}.$$

We can compare with $p\alpha = RT \equiv R_d T_\rho$.

$$p\alpha = R_d T \left(1 + \frac{r_v}{\varepsilon}\right) \frac{1}{1 + r_v + r_l} = R_d T_\rho \quad => \quad T_\rho = \left(1 + \frac{r_v}{\varepsilon}\right) \frac{1}{1 + r_v + r_l} T$$

$$\boxed{T_\rho = \frac{1 + \frac{r_v}{\varepsilon}}{1 + r_v + r_l} T}$$

Introducing $p\alpha = RT \equiv R_d T_\rho$ means that we have an equivalent temperature that dry air would exist at the same total pressure and total specific volume of the moist air, which includes the tiny water droplets. Some authors call this density temperature also the virtual temperature as it is a virtual or equivalent temperature.

I like to call it specifically the density temperature or virtual density temperature to avoid confusion with the usual virtual temperature definition, which is reproduced below.

(27) $\boxed{T_v = \dfrac{1+\dfrac{r_v}{\varepsilon}}{1+r_v}T}$ The virtual temperature for moist air.

We have arrived at our 28th review equation: the virtual density temperature.

(28) $\boxed{T_\rho = \dfrac{1+\dfrac{r_v}{\varepsilon}}{1+r_v+r_l}T}$ The density temperature for moist air with water droplets.