Lapse Rate. Prof. Ruiz (Doc), UNC-Asheville (1978-2021), DoctorPhys on YouTube.
Prerequisites: Chemistry, Physics I, II. Calculus I, II, III. Thermodynamics.

Chapter U. Lapse-Rate Derivation Part 1. In this chapter we sketch out our plan to derive the reversible moist adiabatic lapse rate. In the AMS Glossary, saturation is implied. To emphasize this feature, I will refer to this lapse rate as the

Reversible Saturated Adiabatic Lapse Rate = RSALR.

Over a few chapters we will derive the RSALR formula below, where the AMS Glossary uses the subscript "rm" which stands for reversible moist. But RSALR gives a more precise description: reversible saturated adiabatic lapse rate.

$$\boxed{\Gamma_{rm} = g\frac{(1+r_v+r_l)\left(1+\dfrac{l_v r_v}{R_d T}\right)}{c_{pd}+r_v c_{pv}+r_l c_{pl}+\dfrac{l_v^2 r_v(\varepsilon+r_v)}{R_d T^2}}}$$

I have broken the procedure down into four main steps. Over the next few chapters, if it appears I redo some of the derivations we encountered in the prior review chapters, that is okay. I want the derivation to be as complete as possible. Our goal is calculate the lapse rate $\Gamma_{rm} = -\left(\dfrac{dT}{dz}\right)_{rm}$.

Step A. Entropy $dS = 0$**. Include contributions from dry air, water vapor, and water droplets.**

Step B. Entropy with Mixing Ratios r_v **and** r_l **and** $\dfrac{dS}{m_d} = 0$ **.**

Step C. Solve for $\dfrac{dT}{dp}$ **from** $\dfrac{dS}{m_d} = 0$**.**

Step D. Multiply $\dfrac{dT}{dp}$ **by** $-\dfrac{dp}{dz} = \rho g$**.**

Step E. Rearrange Terms. This step will transform our solution to the above form for Γ_{rm}.

Step A. Entropy. We consider a reversible saturated adiabatic rising air parcel. The parcel includes dry air, water vapor and the presence of small water droplets suspended in the rising moist air due to condensation. Both evaporation and condensation are occurring at the same rate. The respective masses of the dry air, water vapor, and liquid water are m_d, m_v, and m_l. Let the specific entropies of the dry air, water vapor, and liquid water be s_d, s_v, and s_l. The total entropy is given by the sum of the entropies for the dry air $S_d = m_d s_d$, water vapor $S_v = m_v s_v$, and liquid water $S_l = m_l s_l$.

$S = S_d + S_v + S_l$ => $S = m_d s_d + m_v s_v + m_l s_l$ (Our Review Equation 32)

The total entropy S is related to the specific total entropy s through the total mass m:

$S = ms$, where $m = m_d + m_v + m_l$.

For our reversible adiabatic process there is no heat transfer into or out of the system and therefore no change in total entropy. Therefore $dS = \frac{\delta Q}{T} = \frac{m\delta q}{T} = 0$. We can write

$$dS = d(m_d s_d + m_v s_v + m_l s_l) = 0\,.$$

Working through the terms,

$$m_d ds_d + m_v ds_v + m_w ds_w + s_d dm_d + s_v dm_v + s_l dm_l = 0\,.$$

The dry air stays constant, but some vapor can condense to water and some water in turn, can evaporate. However, the sum of the vapor mass and liquid mass is constant, i.e., $d(m_v + m_l) = 0$. Therefore, for the dry air $dm_d = 0$ and for the water (liquid and vapor) $dm_l = -dm_v$. Using these relations,

$m_d ds_d + m_v ds_v + m_w ds_w + s_d dm_d + s_v dm_v + s_l dm_l = 0$ becomes

$$m_d ds_d + m_v ds_v + m_l ds_l + s_v dm_v - s_l dm_v = 0\,,$$

$$m_d ds_d + m_v ds_v + m_l ds_l + (s_v - s_l) dm_v = 0\,.$$

Let's focus on $s_v - s_l$. We can relate this difference to the heat $\delta q = l_v$ needed to go from liquid water to vapor. Since $\delta q = Tds$,

$l_v = \delta q = Tds = T(s_v - s_l)$, which leads to $(s_v - s_l) = \frac{l_v}{T}$.

Our equation $m_d ds_d + m_v ds_v + m_l ds_l + (s_v - s_l) dm_v = 0$

then becomes

$$m_d ds_d + m_v ds_v + m_l ds_l + \frac{l_v}{T} dm_v = 0 .$$

Now we refer to our three equations below.

(29) $ds_d = \frac{c_{pd}}{T} dT - \frac{R_d}{p_d} dp_d$ (30) $ds_v = \frac{c_{pv}}{T} dT - \frac{R_v}{e_s} de_s$ (31) $ds_l = \frac{c_{pl}}{T} dT$

Summary: $\boxed{dS = m_d ds_d + m_v ds_v + m_l ds_l + \frac{l_v}{T} dm_v = 0}$

$$\boxed{ds_d = \frac{c_{pd}}{T} dT - \frac{R_d}{p_d} dp_d} \qquad \boxed{ds_v = \frac{c_{pv}}{T} dT - \frac{R_v}{e_s} de_s} \qquad \boxed{ds_l = \frac{c_{pl}}{T} dT}$$