Lapse Rate. Prof. Ruiz (Doc), UNC-Asheville (1978-2021), DoctorPhys on YouTube.
Prerequisites: Chemistry, Physics I, II. Calculus I, II, III. Thermodynamics.

Chapter V. Lapse-Rate Derivation Part 2. In this chapter we continue our derivation of the Reversible Saturated Adiabatic Lapse Rate - RSALR. We left off in our last chapter with the following summary.

Summary: $$\boxed{dS = m_d ds_d + m_v ds_v + m_l ds_l + \frac{l_v}{T} dm_v = 0}$$

$$\boxed{ds_d = \frac{c_{pd}}{T} dT - \frac{R_d}{p_d} dp_d} \qquad \boxed{ds_v = \frac{c_{pv}}{T} dT - \frac{R_v}{e_s} de_s} \qquad \boxed{ds_l = \frac{c_{pl}}{T} dT}$$

Now it's time to make the three substitutions in $dS = m_d ds_d + m_v ds_v + m_l ds_l + \frac{l_v}{T} dm_v = 0$.

$$dS = m_d ds_d + m_v ds_v + m_l ds_l + \frac{l_v}{T} dm_v = 0 \qquad =>$$

$$dS = m_d \left(\frac{c_{pd}}{T} dT - \frac{R_d}{p_d} dp_d \right) + m_v \left(\frac{c_{pv}}{T} dT - \frac{R_v}{e_s} de_s \right) + m_l \left(\frac{c_{pl}}{T} dT \right) + \frac{l_v}{T} dm_v = 0$$

Rearrange the terms.

$$\boxed{dS = (m_d c_{pd} + m_v c_{pv} + m_l c_{pl}) \frac{dT}{T} - m_d R_d \frac{dp_d}{p_d} - m_v R_v \frac{de_s}{e_s} + \frac{l_v}{T} dm_v = 0}$$

Step B. Entropy with Mixing Ratios. We now divide by the mass of the dry air m_d, a constant.

$$\frac{dS}{m_d} = 0 \qquad => \qquad \left(\frac{m_d}{m_d} c_{pd} + \frac{m_v}{m_d} c_{pv} + \frac{m_l}{m_d} c_{pl}\right) \frac{dT}{T} - \frac{m_d}{m_d} R_d \frac{dp_d}{p_d} - \frac{m_v}{m_d} R_v \frac{de_s}{e_s} + \frac{l_v}{T} \frac{dm_v}{m_d} = 0$$

We are ready to use some of the standard definitions we introduced earlier.

The ratio of vapor to dry air: $r_v \equiv \frac{m_v}{m_d}$, and the ratio of liquid water to dry air: $r_l \equiv \frac{m_l}{m_d}$.

With these substitutions, $\frac{dS}{m_d} = \left(\frac{m_d}{m_d} c_{pd} + \frac{m_v}{m_d} c_{pv} + \frac{m_l}{m_d} c_{pl}\right) \frac{dT}{T} - \frac{m_d}{m_d} R_d \frac{dp_d}{p_d} - \frac{m_v}{m_d} R_v \frac{de_s}{e_s} + \frac{l_v}{T} \frac{dm_v}{m_d} = 0$

becomes

$$\boxed{\frac{dS}{m_d} = (c_{pd} + r_v c_{pv} + r_l c_{pl})\frac{dT}{T} - R_d \frac{dp_d}{p_d} - r_v R_v \frac{de_s}{e_s} + \frac{l_v}{T} dr_v = 0}.$$

In order to save time writing at this point, let's define

$$\boxed{c_p \equiv c_{pd} + r_v c_{pv} + r_l c_{pl}}.$$

Then our above equation $\boxed{\frac{dS}{m_d} = 0}$, becomes

$$\boxed{c_p \frac{dT}{T} - R_\mathrm{d} \frac{dp_\mathrm{d}}{p_\mathrm{d}} - r_\mathrm{v} R_\mathrm{v} \frac{de_\mathrm{s}}{e_\mathrm{s}} + \frac{l_\mathrm{v}}{T} dr_\mathrm{v} = 0}.$$

Step C. Solve for $\frac{dT}{dp}$ from $\frac{dS}{m_d} = 0$. We start with $\boxed{c_p \frac{dT}{T} - R_\mathrm{d} \frac{dp_\mathrm{d}}{p_\mathrm{d}} - r_\mathrm{v} R_\mathrm{v} \frac{de_\mathrm{s}}{e_\mathrm{s}} + \frac{l_\mathrm{v}}{T} dr_\mathrm{v} = 0}$.

The next step is to solve for $\frac{dT}{dp}$ using $c_p \frac{dT}{T} - R_d \frac{dp_d}{p_d} - r_v R_v \frac{de_s}{e_s} + \frac{l_v}{T} dr_v = 0$.

$$c_p \frac{dT}{T} - R_d \frac{dp_d}{p_d} - r_v R_v \frac{de_s}{e_s} + \frac{l_v}{T} dr_v = 0 \quad \Rightarrow \quad \frac{c_p}{T}\frac{dT}{dp} - \frac{R_d}{p_d}\frac{dp_d}{dp} - \frac{r_v R_v}{e_s}\frac{de_s}{dp} + \frac{l_v}{T}\frac{dr_v}{dp} = 0$$

Our aim is to put this equation in the form $a\frac{dT}{dp} + b = 0$, where a and b are functions. We then will have

$$a\frac{dT}{dp} + b = 0 \quad \Rightarrow \quad a\frac{dT}{dp} = -b \quad \Rightarrow \quad \frac{dT}{dp} = -\frac{b}{a}.$$

The final step will be to use the hydrostatic equation $\frac{dp}{dz} = -\rho g$ to obtain $\frac{dT}{dp}\frac{dp}{dz} = \frac{dT}{dz}$.

The left side of our equation $\frac{c_p}{T}\frac{dT}{dp} - \frac{R_d}{p_d}\frac{dp_d}{dp} - \frac{r_v R_v}{e_s}\frac{de_s}{dp} + \frac{l_v}{T}\frac{dr_v}{dp} = 0$ has four terms.

(1) $\frac{c_p}{T}\frac{dT}{dp}$	(2) $-\frac{R_d}{p_d}\frac{dp_d}{dp}$	(3) $-\frac{r_v R_v}{e_s}\frac{de_s}{dp}$	(4) $\frac{l_v}{T}\frac{dr_v}{dp}$

We want to simplify these four terms. We will be happy if we get each of these components to be something times $\frac{dT}{dp}$ or some function with the non-$\frac{dT}{dp}$ derivatives worked out. We will consider each term , one by one. We will conclude this chapter considering Term (1) and Term (2).

Term (1): $\boxed{\frac{c_p}{T}\frac{dT}{dp}}$. This formula is in the form we want since we have some function times $\frac{dT}{dp}$. We are finished for the first term. We have the desired form.

(Term 1) $\boxed{\frac{c_p}{T}\frac{dT}{dp}}$

Term (2): $\boxed{-\frac{R_d}{p_d}\frac{dp_d}{dp}}$. We need to work out the derivative $\frac{dp_d}{dp}$. The partial pressure of the dry air p_{d} is related to the total pressure p as follows: $p = p_{\mathrm{d}} + e_{\mathrm{s}}$. Therefore,

$$\frac{dp_d}{dp} = \frac{d}{dp}(p - e_s) \quad => \quad \frac{dp_d}{dp} = 1 - \frac{de_s}{dp}.$$

Remembering that we are aiming to get $\frac{dT}{dp}$ out of here if we can, we use the chain rule.

$$\frac{dp_d}{dp} = 1 - \frac{de_s}{dT}\frac{dT}{dp}.$$

At this point, we invoke the Clausius-Clapeyron equation $\frac{de_s}{dT} = \frac{l_v e_s}{R_v T^2}$ (24);

$\frac{dp_d}{dp} = 1 - \frac{de_s}{dT}\frac{dT}{dp}$ with $\frac{de_s}{dT} = \frac{l_v e_s}{R_v T^2}$ becomes $\frac{dp_d}{dp} = 1 - \frac{l_v e_s}{R_v T^2}\frac{dT}{dp}$.

We can't forget that the full term is $-\frac{R_d}{p_d}\frac{dp_d}{dp}$, which is then $-\frac{R_d}{p_d}(1 - \frac{l_v e_s}{R_v T^2}\frac{dT}{dp})$.

$$-\frac{R_d}{p_d}(1 - \frac{l_v e_s}{R_v T^2}\frac{dT}{dp}) \quad => \quad -\frac{R_d}{p_d} + \frac{R_d}{p_d}\frac{l_v e_s}{R_v T^2}\frac{dT}{dp}$$

Substitute $p_d = p - e_s$.

$$-\frac{R_d}{p_d}+\frac{R_d}{p_d}\frac{l_v e_s}{R_v T^2}\frac{dT}{dp} \quad => \quad -\frac{R_d}{p-e_s}+\frac{R_d}{p-e_s}\frac{l_v e_s}{R_v T^2}\frac{dT}{dp}$$

At this point, remember that $\frac{R_d}{R_v}=\varepsilon$ (6). Therefore,

$$-\frac{R_d}{p-e_s}+\frac{R_d}{p-e_s}\frac{l_v e_s}{R_v T^2}\frac{dT}{dp} \quad => \quad -\frac{R_d}{p-e_s}+\frac{1}{p-e_s}\frac{R_d}{R_v}\frac{l_v e_s}{T^2}\frac{dT}{dp} \quad => \quad -\frac{R_d}{p-e_s}+\frac{\varepsilon}{p-e_s}\frac{l_v e_s}{T^2}\frac{dT}{dp}$$

$$=> \quad -\frac{R_d}{p-e_s}+\frac{\varepsilon e_s}{p-e_s}\frac{l_v}{T^2}\frac{dT}{dp}$$

We can substitute our review Equation (26) $\boxed{r_v=\frac{\varepsilon e_s}{p-e_s}}$ in $-\frac{R_d}{p-e_s}+\frac{\varepsilon e_s}{p-e_s}\frac{l_v}{T^2}\frac{dT}{dp}$ and obtain

$$-\frac{R_d}{p-e_s}+r_v\frac{l_v}{T^2}\frac{dT}{dp}.$$

We have our second term simplified.

(Term 2) $\boxed{-\frac{R_d}{p-e_s}+\frac{r_v l_v}{T^2}\frac{dT}{dp}}$

We place these results in a table after the original terms that they came from.

	Term 1	**Term 2**	**Term 3**	**Term 4**
Original Terms	$\frac{c_p}{T}\frac{dT}{dp}$	$-\frac{R_d}{p_d}\frac{dp_d}{dp}$	$-\frac{r_v R_v}{e_s}\frac{de_s}{dp}$	$\frac{l_v}{T}\frac{dr_v}{dp}$
Worked Out	$\frac{c_p}{T}\frac{dT}{dp}$	$-\frac{R_d}{p-e_s}+\frac{r_v l_v}{T^2}\frac{dT}{dp}$	To Do.	To Do.