Chapter W. Lapse-Rate Derivation Part 3. We continue with our derivation of the Reversible Saturated Adiabatic Lapse Rate (RSALR). We left off in our last chapter with the following summary table.

	Term 1	Term 2	Term 3	Term 4
Original Terms	$\frac{c_p}{T}\frac{dT}{dp}$	$-\frac{R_d}{p_d}\frac{dp_d}{dp}$	$-\frac{r_v R_v}{e_s}\frac{de_s}{dp}$	$\frac{l_v}{T}\frac{dr_v}{dp}$
Worked Out	$\frac{c_p}{T}\frac{dT}{dp}$	$-\frac{R_d}{p-e_s}+\frac{r_v l_v}{T^2}\frac{dT}{dp}$	To Do.	To Do.

Term (3): $\boxed{-\frac{r_v R_v}{e_s}\frac{de_s}{dp}}$. We already did a trick to deal with $\frac{de_s}{dp}$ using the chain rule : $\frac{de_s}{dp}=\frac{de_s}{dT}\frac{dT}{dp}$. We do it again.

$$-\frac{r_v R_v}{e_s}\frac{de_s}{dp} \quad => \quad -\frac{r_v R_v}{e_s}\left(\frac{de_s}{dT}\right)\frac{dT}{dp} \quad => \quad -\frac{r_v R_v}{e_s}\left(\frac{l_v e_s}{R_v T^2}\right)\frac{dT}{dp} \qquad \text{using (24) } \frac{de_s}{dT}=\frac{l_v e_s}{R_v T^2}$$

$$-\frac{r_v R_v}{e_s}\left(\frac{l_v e_s}{R_v T^2}\right)\frac{dT}{dp} \quad => \quad -r_v\frac{l_v}{T^2}\frac{dT}{dp}$$

$$\text{(Term 3)} \quad \boxed{-\frac{r_v l_v}{T^2}\frac{dT}{dp}} \quad \text{(success)}$$

Term (4): $\boxed{\frac{l_v}{T}\frac{dr_v}{dp}}$. We need to work with $\frac{dr_v}{dp}$.

We calculate the derivative $\frac{dr_v}{dp}$ using review Equation (26) $\boxed{r_v=\frac{\varepsilon e_s}{p-e_s}}$.

$$\frac{dr_v}{dp}=\frac{d}{dp}\left(\frac{\varepsilon e_s}{p-e_s}\right) \quad => \quad \frac{dr_v}{dp}=\frac{\varepsilon}{p-e_s}\frac{de_s}{dp}+\varepsilon e_s\frac{d}{dp}\left(\frac{1}{p-e_s}\right)$$

The second term gives

$$\varepsilon e_s \frac{d}{dp}\left(\frac{1}{p-e_s}\right) = -\varepsilon e_s \frac{1}{\left(p-e_s\right)^2}\frac{d}{dp}(p-e_s) = -\varepsilon e_s \frac{1}{\left(p-e_s\right)^2}\left(1-\frac{de_s}{dp}\right).$$

Together, both terms are now

$$\frac{dr_v}{dp} = \frac{\varepsilon}{p-e_s}\frac{de_s}{dp} - \varepsilon e_s \frac{1}{\left(p-e_s\right)^2}\left(1-\frac{de_s}{dp}\right).$$

We write the two $\frac{de_s}{dp}$ terms first.

$$\frac{dr_v}{dp} = \frac{\varepsilon}{p-e_s}\frac{de_s}{dp} - \varepsilon e_s \frac{1}{\left(p-e_s\right)^2}\left(1-\frac{de_s}{dp}\right) \quad => \quad \frac{dr_v}{dp} = \frac{\varepsilon}{p-e_s}\frac{de_s}{dp} + \varepsilon e_s \frac{1}{\left(p-e_s\right)^2}\frac{de_s}{dp} - \varepsilon e_s \frac{1}{\left(p-e_s\right)^2}$$

$$=> \quad \frac{dr_v}{dp} = \frac{1}{p-e_s}\left(\varepsilon + \frac{\varepsilon e_s}{p-e_s}\right)\frac{de_s}{dp} - \frac{\varepsilon e_s}{\left(p-e_s\right)^2} \quad => \quad \frac{dr_v}{dp} = \frac{1}{p-e_s}\left(\varepsilon + \frac{\varepsilon e_s}{p-e_s}\right)\frac{de_s}{dp} - \frac{1}{p-e_s}\frac{\varepsilon e_s}{p-e_s}$$

At this point we use again review Equation (26) $\boxed{r_v = \frac{\varepsilon e_s}{p-e_s}}$. Therefore,

$$\frac{dr_v}{dp} = \frac{1}{p-e_s}\left(\varepsilon + \frac{\varepsilon e_s}{p-e_s}\right)\frac{de_s}{dp} - \frac{1}{p-e_s}\frac{\varepsilon e_s}{p-e_s} \quad => \quad \frac{dr_v}{dp} = \frac{1}{p-e_s}(\varepsilon + r_v)\frac{de_s}{dp} - \frac{r_v}{p-e_s}$$

$$\frac{dr_v}{dp} = \left(\frac{\varepsilon + r_v}{p-e_s}\right)\frac{de_s}{dp} - \frac{r_v}{p-e_s}.$$

But since we need to flush out $\frac{dT}{dp}$ terms, we use our chain-rule trick $\frac{de_s}{dp}$ as $\frac{de_s}{dT}\frac{dT}{dp}$.

$$\frac{dr_v}{dp} = \left(\frac{\varepsilon + r_v}{p-e_s}\right)\frac{de_s}{dT}\frac{dT}{dp} - \frac{r_v}{p-e_s}$$

At this point the Clausius-Clapeyron equation (24) $\frac{de_s}{dT} = \frac{e_s l_v}{R_v T^2}$ beckons.

Substituting the Clausius-Clapeyron equation brings us to

$$\frac{dr_v}{dp} = \left(\frac{\varepsilon + r_v}{p - e_s}\right)\frac{e_s l_v}{R_v T^2}\frac{dT}{dp} - \frac{r_v}{p - e_s} .$$

Finally, we recall that Term 4 is $\frac{l_v}{T}\frac{dr_v}{dp}$. We need to include the $\frac{l_v}{T}$ factors for each term.

$$\frac{l_v}{T}\frac{dr_v}{dp} = \frac{l_v}{T}\left(\frac{\varepsilon + r_v}{p - e_s}\right)\frac{e_s l_v}{R_v T^2}\frac{dT}{dp} - \frac{l_v}{T}\frac{r_v}{p - e_s}$$

We rearrange things slightly to obtain

$$\frac{l_v}{T}\frac{dr_v}{dp} = \left(\frac{\varepsilon + r_v}{p - e_s}\right)\frac{e_s l_v^2}{R_v T^3}\frac{dT}{dp} - \frac{1}{p - e_s}\frac{r_v l_v}{T} .$$

(Term 4) $\boxed{\left(\frac{\varepsilon + r_v}{p - e_s}\right)\frac{e_s l_v^2}{R_v T^3}\frac{dT}{dp} - \frac{1}{p - e_s}\frac{r_v l_v}{T}}$ (success)

Summary of Our Four Terms worked out:

$$\boxed{\frac{c_p}{T}\frac{dT}{dp} - \frac{R_d}{p_d}\frac{dp_d}{dp} - \frac{r_v R_v}{e_s}\frac{de_s}{dp} + \frac{l_v}{T}\frac{dr_v}{dp} = 0}$$

(Term 1) $\boxed{\frac{c_p}{T}\frac{dT}{dp}}$ (Term 2) $\boxed{-\frac{R_d}{p - e_s} + \frac{r_v l_v}{T^2}\frac{dT}{dp}}$

(Term 3) $\boxed{-\frac{r_v l_v}{T^2}\frac{dT}{dp}}$ (Term 4) $\boxed{\left(\frac{\varepsilon + r_v}{p - e_s}\right)\frac{e_s l_v^2}{R_v T^3}\frac{dT}{dp} - \frac{1}{p - e_s}\frac{r_v l_v}{T}}$

We place these results in a table after the original terms that they came from.

	Term 1	**Term 2**	**Term 3**	**Term 4**
Original Terms	$\frac{c_p}{T}\frac{dT}{dp}$	$-\frac{R_d}{p_d}\frac{dp_d}{dp}$	$-\frac{r_v R_v}{e_s}\frac{de_s}{dp}$	$\frac{l_v}{T}\frac{dr_v}{dp}$
Worked Out	$\frac{c_p}{T}\frac{dT}{dp}$	$-\frac{R_d}{p - e_s} + \frac{r_v l_v}{T^2}\frac{dT}{dp}$	$-\frac{r_v l_v}{T^2}\frac{dT}{dp}$	$\left(\frac{\varepsilon + r_v}{p - e_s}\right)\frac{e_s l_v^2}{R_v T^3}\frac{dT}{dp} - \frac{1}{p - e_s}\frac{r_v l_v}{T}$