Lapse Rate. Prof. Ruiz (Doc), UNC-Asheville (1978-2021), DoctorPhys on YouTube.
Prerequisites: Chemistry, Physics I, II. Calculus I, II, III. Thermodynamics.

Chapter X. Lapse-Rate Derivation Part 4. We continue with our derivation of the Reversible Saturated Adiabatic Lapse Rate (RSALR). We left off in our last chapter with the following summary table.

	Term 1	Term 2	Term 3	Term 4
Original Terms	$\frac{c_p}{T}\frac{dT}{dp}$	$-\frac{R_d}{p_d}\frac{dp_d}{dp}$	$-\frac{r_v R_v}{e_s}\frac{de_s}{dp}$	$\frac{l_v}{T}\frac{dr_v}{dp}$
Worked Out	$\frac{c_p}{T}\frac{dT}{dp}$	$-\frac{R_d}{p-e_s}+\frac{r_v l_v}{T^2}\frac{dT}{dp}$	$-\frac{r_v l_v}{T^2}\frac{dT}{dp}$	$\left(\frac{\varepsilon+r_v}{p-e_s}\right)\frac{e_s l_v^2}{R_v T^3}\frac{dT}{dp}-\frac{1}{p-e_s}\frac{r_v l_v}{T}$

The next step is to separate out the $\frac{dT}{dp}$ components. We go back to our aim to find a formula of the form

$$a\frac{dT}{dp}+b=0$$

in order to obtain $\frac{dT}{dp}=-\frac{b}{a}$. We separate the $\frac{dT}{dp}$ terms from the rest in our work table below.

Work Table

$a\frac{dT}{dp}+b$	$\frac{c_p}{T}\frac{dT}{dp}-\frac{R_d}{p_d}\frac{dp_d}{dp}-\frac{r_v R_v}{e_s}\frac{de_s}{dp}+\frac{l_v}{T}\frac{dr_v}{dp}=0\equiv a\frac{dT}{dp}+b$			
	Term 1	Term 2	Term 3	Term 4
Original Terms	$\frac{c_p}{T}\frac{dT}{dp}$	$-\frac{R_d}{p_d}\frac{dp_d}{dp}$	$-\frac{r_v R_v}{e_s}\frac{de_s}{dp}$	$\frac{l_v}{T}\frac{dr_v}{dp}$
Worked Out	$\frac{c_p}{T}\frac{dT}{dp}$	$-\frac{R_d}{p-e_s}+\frac{r_v l_v}{T^2}\frac{dT}{dp}$	$-\frac{r_v l_v}{T^2}\frac{dT}{dp}$	$\left(\frac{\varepsilon+r_v}{p-e_s}\right)\frac{e_s l_v^2}{R_v T^3}\frac{dT}{dp}-\frac{1}{p-e_s}\frac{r_v l_v}{T}$
a **Terms**	$\frac{c_p}{T}$	$\frac{r_v l_v}{T^2}$	$-\frac{r_v l_v}{T^2}$	$\left(\frac{\varepsilon+r_v}{p-e_s}\right)\frac{e_s l_v^2}{R_v T^3}$
b **Terms**	None	$-\frac{R_d}{p-e_s}$	None	$-\frac{1}{p-e_s}\frac{r_v l_v}{T}$

$$a=\frac{c_p}{T}+\frac{r_v l_v}{T^2}-\frac{r_v l_v}{T^2}+\left(\frac{\varepsilon+r_v}{p-e_s}\right)\frac{e_s l_v^2}{R_v T^3} \qquad b=-\frac{R_d}{p-e_s}-\frac{1}{p-e_s}\frac{r_v l_v}{T}$$

We find a nice cancellation of two terms in a.

$$a = \frac{c_p}{T} + \left(\frac{\varepsilon + r_v}{p - e_s}\right)\frac{e_s l_v^2}{R_v T^3} \qquad b = -\frac{R_d}{p - e_s} - \frac{1}{p - e_s}\frac{r_v l_v}{T}$$

We can now write our derivative $\frac{dT}{dp} = -\frac{b}{a}$.

$$\frac{dT}{dp} = -\frac{b}{a} \quad \text{with} \quad a = \frac{c_p}{T} + \left(\frac{\varepsilon + r_v}{p - e_s}\right)\frac{e_s l_v^2}{R_v T^3} \quad \text{and} \quad b = -\frac{R_d}{p - e_s} - \frac{1}{p - e_s}\frac{r_v l_v}{T},$$

$$\frac{dT}{dp} = -\frac{b}{a} \quad \Rightarrow \quad \frac{dT}{dp} = -\frac{\left[-\frac{R_d}{p - e_s} - \frac{1}{p - e_s}\frac{r_v l_v}{T}\right]}{\frac{c_p}{T} + \left(\frac{\varepsilon + r_v}{p - e_s}\right)\frac{e_s l_v^2}{R_v T^3}} \quad \Rightarrow \quad \frac{dT}{dp} = \frac{\frac{R_d}{p - e_s} + \frac{1}{p - e_s}\frac{r_v l_v}{T}}{\frac{c_p}{T} + \left(\frac{\varepsilon + r_v}{p - e_s}\right)\frac{e_s l_v^2}{R_v T^3}}$$

$$\boxed{\frac{dT}{dp} = \frac{\frac{1}{p - e_s}\left(R_d + \frac{r_v l_v}{T}\right)}{\frac{c_p}{T} + \left(\frac{\varepsilon + r_v}{p - e_s}\right)\frac{e_s l_v^2}{R_v T^3}}}$$

Remember that $c_p \equiv c_{pd} + r_v c_{pv} + r_l c_{pl}$. Later we will substitute it in.

Step D. Multiply $\frac{dT}{dp}$ by $-\frac{dp}{dz} = \rho g$. The last step is to find the reversible adiabatic moist lapse rate $\Gamma_{rm} = -\frac{dT}{dz}$ using $\frac{dT}{dp}\frac{dp}{dz}$. The hydrostatic equation $\frac{dp}{dz} = -\rho g$ is needed at this point. But what is the density ρ. We need to proceed cautiously since water droplets are present and the ideal gas idea does not apply there. The air parcel consists of dry air, water vapor, and water droplets.

We first start with the definition of the density. The density is mass divided by volume. Take the volume of an air parcel containing dry air, vapor, and water droplets to be V. The respective masses of the dry air, vapor, and liquid droplets are m_d, m_v, and m_l. The density is then given by the total mass divided by the volume.

$$\rho = \frac{m_d + m_v + m_l}{V}$$

Factor out $\frac{m_d}{V}$.

$$\rho = \frac{m_d + m_v + m_l}{V} \quad => \quad \rho = \frac{1}{V}(m_d + m_v + m_l) \quad => \quad \rho = \frac{m_d}{V}\left(1 + \frac{m_v}{m_d} + \frac{m_l}{m_d}\right)$$

Now introduce the mixing ratios $r_v = \frac{m_v}{m_d}$ and $r_l = \frac{m_l}{m_d}$.

$$\rho = \frac{m_d}{V}\left(1 + \frac{m_v}{m_d} + \frac{m_l}{m_d}\right) \quad => \quad \rho = \frac{m_d}{V}(1 + r_v + r_l)$$

The factor $\frac{m_d}{V}$ is the density of the dry air ρ_d.

$$\rho = \frac{m_d}{V}(1 + r_v + r_l) \quad => \quad \rho = \rho_d (1 + r_v + r_l)$$

The total pressure is given by the sum of the pressures of dry air and vapor. The liquid water droplets do not add to the pressure.

$$p = p_d + e$$

So far we have not used the ideal gas law. However, we can now do so for the dry air and vapor.

$$p_d = \rho_d R_d T \qquad e = \rho_v R_v T$$

We insert these into the equation for the total pressure: $p = p_d + e$ => $p = \rho_d R_d T + \rho_v R_v T$.

Use $\varepsilon = \frac{R_d}{R_v}$ in the form of $R_v = \frac{R_d}{\varepsilon}$ to eliminate R_v.

$$p = \rho_d R_d T + \rho_v R_v T \quad => \quad p = \rho_d R_d T + \rho_v \frac{R_d}{\varepsilon} T$$

Factor out ρ_d: $p = \rho_d R_d T + \rho_v \frac{R_d}{\varepsilon} T$ => $p = \rho_d \left(R_d T + \frac{\rho_v}{\rho_d} \frac{R_d}{\varepsilon} T \right)$.

$$p = \rho_d \left(R_d T + \frac{\rho_v}{\rho_d} \frac{R_d}{\varepsilon} T \right) \quad => \quad p = \rho_d R_d T \left(1 + \frac{\rho_v}{\rho_d} \frac{1}{\varepsilon} \right)$$

Note that the mixing ratio $r_v = \frac{m_v}{m_d}$ is also $r_v = \frac{\rho_v}{\rho_d}$.

$$p = \rho_d R_d T\left(1 + \frac{\rho_v}{\rho_d}\frac{1}{\varepsilon}\right) \quad \text{=>} \quad p = \rho_d R_d T\left(1 + \frac{r_v}{\varepsilon}\right)$$

From $\rho = \rho_d\left(1 + r_v + r_l\right)$ we have $\rho_d = \dfrac{\rho}{1 + r_v + r}$.

$$p = \rho_d R_d T\left(1 + \frac{r_v}{\varepsilon}\right) \quad \text{=>} \quad p = \frac{\rho}{1 + r_v + r} R_d T\left(1 + \frac{r_v}{\varepsilon}\right)$$

Solve for the density ρ.

$$p = \frac{\rho}{1 + r_v + r} R_d T\left(1 + \frac{r_v}{\varepsilon}\right) \quad \text{=>} \quad \left(1 + r_v + r\right)p = \rho R_d T\left(1 + \frac{r_v}{\varepsilon}\right) \quad \text{=>} \quad \rho = \frac{\left(1 + r_v + r_l\right)p}{R_d T\left(1 + \dfrac{r_v}{\varepsilon}\right)}$$

$$\boxed{\rho = \frac{p}{R_d T}\frac{\left(1 + r_v + r_l\right)}{\left(1 + \dfrac{r_v}{\varepsilon}\right)}}$$

Combining $\dfrac{dp}{dz} = -\rho g$ and $\rho = \dfrac{p}{R_d T}\dfrac{1 + r_v + r_l}{1 + \dfrac{r_v}{\varepsilon}}$ results in

$$\boxed{\frac{dp}{dz} = -\frac{p}{R_d T}\frac{1 + r_v + r_l}{1 + \dfrac{r_v}{\varepsilon}}g}$$

Summary.

$$\boxed{\frac{dT}{dp} = \frac{\dfrac{1}{p - e_s}\left(R_d + \dfrac{r_v l_v}{T}\right)}{\dfrac{c_p}{T} + \left(\dfrac{\varepsilon + r_v}{p - e_s}\right)\dfrac{e_s l_v^2}{R_v T^3}}} \qquad \boxed{\frac{dp}{dz} = -\frac{p}{R_d T}\frac{1 + r_v + r_l}{1 + \dfrac{r_v}{\varepsilon}}g}$$

$$\boxed{\Gamma_{rm} = -\frac{dT}{dz} = -\frac{dT}{dp}\frac{dp}{dz} = -\left[\frac{\dfrac{1}{p - e_s}\left(R_d + \dfrac{r_v l_v}{T}\right)}{\dfrac{c_p}{T} + \left(\dfrac{\varepsilon + r_v}{p - e_s}\right)\dfrac{e_s l_v^2}{R_v T^3}}\right]\left(-\frac{p}{R_d T}\frac{1 + r_v + r_l}{1 + \dfrac{r_v}{\varepsilon}}g\right)}$$