Lapse Rate. Prof. Ruiz (Doc), UNC-Asheville (1978-2021), DoctorPhys on YouTube.
Prerequisites: Chemistry, Physics I, II. Calculus I, II, III. Thermodynamics.

Chapter Y. Lapse-Rate Derivation Part V. We continue with our derivation of the Reversible Saturated Adiabatic Lapse Rate - RSALR. We left off in our last chapter with the following equation.

$$\boxed{\Gamma_{rm} = -\frac{dT}{dz} = -\frac{dT}{dp}\frac{dp}{dz} = -\left[\frac{\frac{1}{p-e_s}\left(R_d + \frac{r_v l_v}{T}\right)}{\frac{c_p}{T} + \left(\frac{\varepsilon + r_v}{p-e_s}\right)\frac{e_s l_v^2}{R_v T^3}}\right]\left(-\frac{p}{R_d T}\frac{1+r_v+r_l}{1+\frac{r_v}{\varepsilon}}g\right)}$$

Step E. Rearrange Terms. The remaining steps consist of algebraic manipulations.

Cancel the signs and move the far right parenthetical factor to the front.	$\Gamma_{rm} = \left(\frac{p}{R_d T}\right)\left(\frac{1+r_v+r_l}{1+\frac{r_v}{\varepsilon}}g\right)\frac{\frac{1}{p-e_s}\left(R_d + \frac{r_v l_v}{T}\right)}{\frac{c_p}{T} + \left(\frac{\varepsilon + r_v}{p-e_s}\right)\frac{e_s l_v^2}{R_v T^3}}$
Separate and move the factors $\frac{p}{R_d}$ and $\frac{1}{T}$.	$\Gamma_{rm} = \left(\frac{1+r_v+r_l}{1+\frac{r_v}{\varepsilon}}g\right)\frac{1}{T}\frac{\frac{p}{R_d}\frac{1}{p-e_s}\left(R_d + \frac{r_v l_v}{T}\right)}{\frac{c_p}{T} + \left(\frac{\varepsilon + r_v}{p-e_s}\right)\frac{e_s l_v^2}{R_v T^3}}$
Separate the factors p and $\frac{1}{R_d}$.	$\Gamma_{rm} = \left(\frac{1+r_v+r_l}{1+\frac{r_v}{\varepsilon}}\right)g\frac{\frac{p}{p-e_s}\frac{1}{R_d}\left(R_d + \frac{r_v l_v}{T}\right)}{T\left[\frac{c_p}{T} + \left(\frac{\varepsilon + r_v}{p-e_s}\right)\frac{e_s l_v^2}{R_v T^3}\right]}$
Replace p with $p_d + e_s$ and $p - e_s$ with p_d in $\frac{p}{p-e_s}$ to get $\frac{p_d+e_s}{p_d}$.	$\Gamma_{rm} = \left(\frac{1+r_v+r_l}{1+\frac{r_v}{\varepsilon}}\right)g\frac{\frac{p_d+e_s}{p_d}\frac{1}{R_d}\left(R_d + \frac{r_v l_v}{T}\right)}{T\left[\frac{c_p}{T} + \left(\frac{\varepsilon + r_v}{p_d}\right)\frac{e_s l_v^2}{R_v T^3}\right]}$
Replace $\frac{p_d+e_s}{p_d}$ with $1+\frac{e_s}{p_d}$.	$\Gamma_{rm} = \left(\frac{1+r_v+r_l}{1+\frac{r_v}{\varepsilon}}\right)g\frac{\left(1+\frac{e_s}{p_d}\right)\frac{1}{R_d}\left(R_d + \frac{r_v l_v}{T}\right)}{T\left[\frac{c_p}{T} + \left(\varepsilon + r_v\right)\frac{e_s}{p_d}\frac{l_v^2}{R_v T^3}\right]}$

Replace $1+\frac{e_s}{p_d}$ with $1+\frac{r_v}{\varepsilon}$ since (25) $\boxed{r_v=\frac{\varepsilon e}{p_d}}$ indicates $\frac{e_s}{p_d}=\frac{r_v}{\varepsilon}$.	$\Gamma_{rm}=\left(\frac{1+r_v+r_l}{1+\frac{r_v}{\varepsilon}}\right)g\frac{\left(1+\frac{r_v}{\varepsilon}\right)\frac{1}{R_d}\left(R_d+\frac{r_v l_v}{T}\right)}{T\left[\frac{c_p}{T}+\left(\varepsilon+r_v\right)\frac{r_v}{\varepsilon}\frac{l_v^2}{R_v T^3}\right]}$
Recognize that $\frac{\left(1+\frac{r_v}{\varepsilon}\right)}{\left(1+\frac{r_v}{\varepsilon}\right)}=1$.	$\Gamma_{rm}=\left(1+r_v+r_l\right)g\frac{\frac{1}{R_d}\left(R_d+\frac{r_v l_v}{T}\right)}{T\left[\frac{c_p}{T}+\left(\varepsilon+r_v\right)\frac{r_v}{\varepsilon}\frac{l_v^2}{R_v T^3}\right]}$
Move $\frac{1}{R_d}$ into $\left(R_d+\frac{r_v l_v}{T}\right)$.	$\Gamma_{rm}=g\frac{\left(1+r_v+r_l\right)\left(1+\frac{r_v l_v}{R_d T}\right)}{T\left[\frac{c_p}{T}+\left(\varepsilon+r_v\right)\frac{r_v}{\varepsilon}\frac{l_v^2}{R_v T^3}\right]}$
Multiply the denominator through by T.	$\Gamma_{rm}=g\frac{\left(1+r_v+r_l\right)\left(1+\frac{r_v l_v}{R_d T}\right)}{c_p+\left(\varepsilon+r_v\right)\frac{r_v}{\varepsilon}\frac{l_v^2}{R_v T^2}}$
Since, $\varepsilon=\frac{R_d}{R_v}$, we have $\varepsilon R_v=R_d$. Then $\frac{r_v}{\varepsilon}\frac{l_v^2}{R_v T^2}=\frac{r_v l_v^2}{R_d T^2}=\frac{l_v^2 r_v}{R_d T^2}$.	$\Gamma_{rm}=g\frac{\left(1+r_v+r_l\right)\left(1+\frac{l_v r_v}{R_d T}\right)}{c_p+\left(\varepsilon+r_v\right)\frac{l_v^2 r_v}{R_d T^2}}$
Note that $\left(\varepsilon+r_v\right)\frac{l_v^2 r_v}{R_d T^2}=\frac{l_v^2 r_v\left(\varepsilon+r_v\right)}{R_d T^2}$.	$\Gamma_{rm}=g\frac{\left(1+r_v+r_l\right)\left(1+\frac{l_v r_v}{R_d T}\right)}{c_p+\frac{l_v^2 r_v\left(\varepsilon+r_v\right)}{R_d T^2}}$

The final step is to substitute in $c_p=c_{pd}+r_v c_{pv}+r_l c_{pl}$.

$$\boxed{\Gamma_{rm}=g\frac{\left(1+r_v+r_l\right)\left(1+\frac{l_v r_v}{R_d T}\right)}{c_{pd}+r_v c_{pv}+r_l c_{pl}+\frac{l_v^2 r_v\left(\varepsilon+r_v\right)}{R_d T^2}}}$$

As we have mentioned earlier, the AMS Glossary uses the additional definition $r_t=r_v+r_l$. I personally like to write out $r_t=r_v+r_l$ rather than introduce another mixing ratio called total.