Lapse Rate. Prof. Ruiz (Doc), UNC-Asheville (1978-2021), DoctorPhys on YouTube.
Prerequisites: Chemistry, Physics I, II. Calculus I, II, III. Thermodynamics.

Chapter Z. Lapse Rates . We arrive at approximate lapse rates from our general RSALR.

1. The Reversible Saturated Adiabatic Lapse Rate (RSALR) Γ_{rm} **.** This lapse rate is the one we have derived. The AMS Glossary uses "rm" for reversible moist. While in general, "m" for moist does not necessarily mean a saturated condition, in this context saturation it is assumed. For this reason I prefer the designation RSALR, since this description spells out Reversible Saturated Adiabatic Lapse Rate.

$$\boxed{\Gamma_{rm} = g\frac{\left(1+r_v+r_l\right)\left(1+\dfrac{l_v r_v}{R_d T}\right)}{c_{pd}+r_v c_{pv}+r_l c_{pl}+\dfrac{l_v^2 r_v\left(\varepsilon+r_v\right)}{R_d T^2}}}$$

2. The Pseudoadiabatic Saturated Adiabatic Lapse Rate (PSALR) Γ_{ps} **.** This lapse rate is obtained from Γ_{rm} by neglecting r_l . We might say that in this approximation as soon as the water droplets form, they drop out of the cloud and we neglect them in our thermodynamic booking. Therefore, we take $r_l = 0$ in Γ_{rm} . The glossary calls the result Γ_{ps} . This process is irreversible.

$$\boxed{\Gamma_{ps} = g\frac{\left(1+r_v\right)\left(1+\dfrac{l_v r_v}{R_d T}\right)}{c_{pd}+r_v c_{pv}+\dfrac{l_v^2 r_v\left(\varepsilon+r_v\right)}{R_d T^2}}}$$

3. The Moist Saturated Adiabatic Lapse Rate (MSALR) or Moist Adiabatic Lapse Rate Γ_m **.** For this lapse rate we make the additional approximation that since r_v is small, we can neglect it when is stands next to much larger items. The typical situation is that $r_v \le 0.02$. Therefore, the following two inequalities are true.

$$r_v << 1 \qquad r_v << \varepsilon = 0.622$$

There is one more relevant inequality.

$$r_v c_{pv} \le 0.02 c_{pv} = 0.02\cdot 1850 \ \frac{\text{J}}{\text{kg}\cdot\text{K}} = 37\frac{\text{J}}{\text{kg}\cdot\text{K}} << c_{pd} = 1005 \ \frac{\text{J}}{\text{kg}\cdot\text{K}}$$

The associated approximations are

$$1+r_v \to 1, \quad \varepsilon + r_v \to \varepsilon, \quad \text{and} \quad c_{pd}+r_v c_{pv} \to c_{pd} .$$

Using these approximations,

$$\Gamma_{ps} = g\frac{(1+r_v)\left(1+\dfrac{l_v r_v}{R_d T}\right)}{c_{pd} + r_v c_{pv} + \dfrac{l_v^2 r_v(\varepsilon + r_v)}{R_d T^2}} \quad => \quad \boxed{\Gamma_m = g\frac{1+\dfrac{l_v r_v}{R_d T}}{c_{pd} + \dfrac{l_v^2 r_v \varepsilon}{R_d T^2}}}$$

Many authors factor out c_{pd}.

$$\boxed{\Gamma_m = \frac{g}{c_{pd}}\frac{1+\dfrac{l_v r_v}{R_d T}}{1+\dfrac{\varepsilon l_v^2 r_v}{c_{pd} R_d T^2}}}$$

They do this factoring because the dry adiabatic lapse rate, as we will see below is $\dfrac{g}{c_{pd}}$.

Since Γ_{ps} and Γ_m are very closely related, meteorologists "tend to use the terms *saturated adiabatic*, *moist adiabatic* and *pseudoadiabatic* more or less interchangeably when applied to rising air parcels" (Petty, 2008). The above moist adiabatic lapse rate is the popular formula derived in undergraduate courses on atmospheric thermodynamics.

4. The Moist Unsaturated Adiabatic Lapse Rate (MUSALR) $\Gamma_{m,unsat}$. There is no condensation in this case since we have not reached saturation. We take $l_v \to 0$ in the pseudoadiabatic saturated lapse rate.

$$\Gamma_{ps} = g\frac{(1+r_v)\left(1+\dfrac{l_v r_v}{R_d T}\right)}{c_{pd} + r_v c_{pv} + \dfrac{l_v^2 r_v(\varepsilon + r_v)}{R_d T^2}} \quad => \quad \boxed{\Gamma_{m,unsat} = g\frac{1+r_v}{c_{pd} + r_v c_{pv}}}$$

5. The Dry Adiabatic Lapse Rate (DALR) Γ_d. For the dry adiabatic lapse rate, there is no water vapor present, i.e., $r_v = 0$.

Our result $\boxed{\Gamma_{m,unsat} = g\dfrac{1+r_v}{c_{pd} + r_v c_{pv}}}$ with $r_v = 0$ becomes $\boxed{\Gamma_d = \dfrac{g}{c_{pd}}}$.

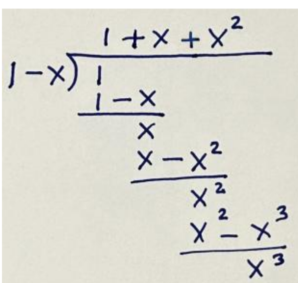
Before we leave this section, I would like to show that our $\Gamma_{m,unsat} = g\dfrac{1+r_v}{c_{pd} + r_v c_{pv}}$ agrees with the formula found in the [AMS Glossary](), which gives the factor $\dfrac{1+r_v}{1+1.85r_v} \approx 1 - 0.85r_v$. Start first with

$$\Gamma_{m,unsat} = g\frac{1+r_v}{c_{pd} + r_v c_{pv}} = \frac{g}{c_{pd}}\frac{1+r_v}{\left(1 + r_v \dfrac{c_{pv}}{c_{pd}}\right)} = \frac{g}{c_{pd}}\frac{1+r_v}{\left(1 + r_v \dfrac{1860 \dfrac{\mathrm{J}}{\mathrm{kg \cdot K}}}{1005 \dfrac{\mathrm{J}}{\mathrm{kg \cdot K}}}\right)} = \frac{g}{c_{pd}}\frac{1+r_v}{1+1.85r_v}.$$

Since $1.85r_v = 1.85 \cdot 0.02 = 0.037$ is small we use an approximation trick that is very useful.

For small x:

$$\frac{1}{1+x} \approx 1 - x \quad \text{and} \quad \frac{1}{1-x} \approx 1 + x.$$



I used to show this approximation to students using long division, but I am not sure they learn long division in grade school anymore. The alternative way is to use the Taylor series expansion, dropping x^2 and high powers as they are extremely small compared to x.

$$f(x) = \frac{1}{1-x} = \sum_n \frac{f^{(n)}(0)}{n!}x^n = 1 + x + x^2 + x^3 + \ldots$$

$$\Gamma_{m,unsat} = \frac{g}{c_{pd}}\frac{1+r_v}{1+1.85r_v} = \frac{g}{c_{pd}}(1+r_v)(1-1.85r_v)$$

Multiply $(1+r_v)(1-1.85r_v)$ through.

$$\Gamma_{m,unsat} = \frac{g}{c_{pd}}(1 + r_v - 1.85r_v - 1.85r_v^2)$$

The quadratic term is extremely small. Therefore, we find the following.

$$\Gamma_{m,unsat} = \frac{g}{c_{pd}}\frac{1+r_v}{1+1.85r_v} = \frac{g}{c_{pd}}(1 + r_v - 1.85r_v) \quad => \quad \Gamma_{m,unsat} = \frac{g}{c_{pd}}\left[1 + (1 - 1.85)r_v\right]$$

$$\Rightarrow \quad \Gamma_{m,unsat} = \frac{g}{c_{pd}}\left[1+(-0.85)r_v\right]$$

$$\Gamma_{m,unsat} = \frac{g}{c_{pd}}\frac{1+r_v}{1+1.85r_v} \approx \frac{g}{c_{pd}}(1-0.85r_v)$$

The above equation is the one found in the AMS Glossary, where it states that the "adiabatic lapse rate of unsaturated air containing water vapor … differs from" $\frac{g}{c_{pd}}$ "by the factor

$$\frac{1+r_v}{1+1.85r_v} \approx 1-0.85r_v,$$

where r_v is the mixing ratio of water vapor and c_{pv} is the specific heat of water vapor."

Summary

$$\Gamma_{rm} = g\frac{(1+r_v+r_l)\left(1+\frac{l_v r_v}{R_d T}\right)}{c_{pd}+r_v c_{pv}+r_l c_{pl}+\frac{l_v^2 r_v(\varepsilon+r_v)}{R_d T^2}}$$

Reversible Saturated Adiabatic Lapse Rate

RSALR

↓

$$\Gamma_{ps} = g\frac{(1+r_v)\left(1+\frac{l_v r_v}{R_d T}\right)}{c_{pd}+r_v c_{pv}+\frac{l_v^2 r_v(\varepsilon+r_v)}{R_d T^2}}$$

Pseudoadiabatic Saturated Lapse Rate

PSALR

↓

$$\Gamma_m = \frac{g}{c_{pd}}\frac{1+\frac{l_v r_v}{R_d T}}{1+\frac{\varepsilon l_v^2 r_v}{c_{pd} R_d T^2}} \longrightarrow \Gamma_{m,unsat} = g\frac{1+r_v}{c_{pd}+r_v c_{pv}} \longrightarrow \Gamma_d = \frac{g}{c_{pd}}$$

Moist Unsaturated Adiabatic Lapse Rate

MUALR

Dry Adiabatic Lapse Rate

DALR

Moist Saturated Adiabatic Lapse Rate

MSALR

For our final summary, below we include the five lapse rates we have derived, arranged from the simplest to the most general and complex.

$$\boxed{\Gamma_d = \frac{g}{c_{pd}}} \qquad \boxed{\Gamma_{m,unsat} = g\frac{1+r_v}{c_{pd}+r_v c_{pv}}}$$

$$\boxed{\Gamma_m = \frac{g}{c_{pd}}\frac{1+\dfrac{l_v r_v}{R_d T}}{1+\dfrac{\varepsilon l_v^2 r_v}{c_{pd} R_d T^2}}} \qquad \boxed{\Gamma_{ps} = g\frac{(1+r_v)\left(1+\dfrac{l_v r_v}{R_d T}\right)}{c_{pd}+r_v c_{pv}+\dfrac{l_v^2 r_v(\varepsilon+r_v)}{R_d T^2}}}$$

$$\boxed{\Gamma_{rm} = g\frac{(1+r_v+r_l)\left(1+\dfrac{l_v r_v}{R_d T}\right)}{c_{pd}+r_v c_{pv}+r_l c_{pl}+\dfrac{l_v^2 r_v(\varepsilon+r_v)}{R_d T^2}}}$$

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