A0. Introduction

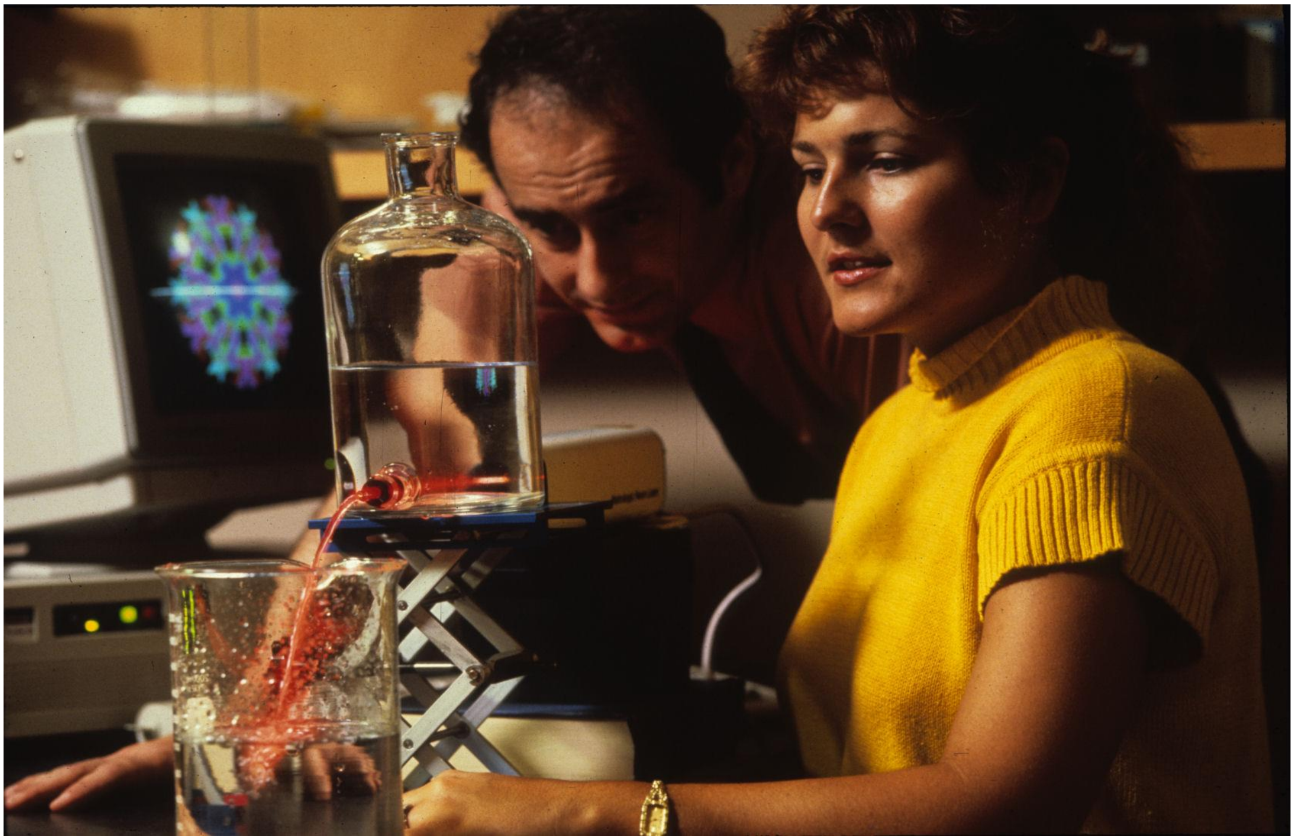


Fig. 1. Total Internal Reflection. Doc and UNCA Student Long Ago.
Credit: Asheville Photographer John Warner
Do you recognize the XT Computer of 1980s?

Light is one of the basic topics in physics and is important in virtually every other discipline. Biology studies organs of sight and the interaction of light with the human eye is rich in chemistry. Then there is the perception of light, the subject of psychology, where both the eye and the brain play essential roles. Perceptual studies led to the application of rendering color in movies, television, and computer monitors. Color is of fundamental importance in art and so on. While light has all these interdisciplinary connections, we will focus on light from the point of view of physics.

See Fig. 1 for a fun demonstration with light where water acts as a light pipe as the red laser light undergoes total internal reflection. The young lady Gwen was a UNCA student and former Office Assistant of our Environmental Studies Program, which later became a Department. The artistic photographer John Warner, my former neighbor, arranged the composition and took the photo.

There are many excellent texts on optics available. One is by my esteemed colleague and neighbor Prof. Charles Bennett, who last taught this course and has retired at the end of the Spring 2020 semester. He was actually scheduled to teach the Fall 2020 course on optics. Another fine text is by Eugene Hecht, who visited UNCA years ago and both Professor Bennett and I got to know Prof. Hecht and his wife. We will use my own materials since my personal philosophy of teaching has been, like Professors Bennett and Hecht, to present physics integrated in one's own personal way. Optics can be broken down into three main areas:

1. Geometrical Optics – light travels in straight lines,
2. Physical Optics – light as waves,
3. Quantum Optics – light as particles (quanta).

Our course description for *PHYS 323 Modern Optics* (3) in the UNCA Catalog is:

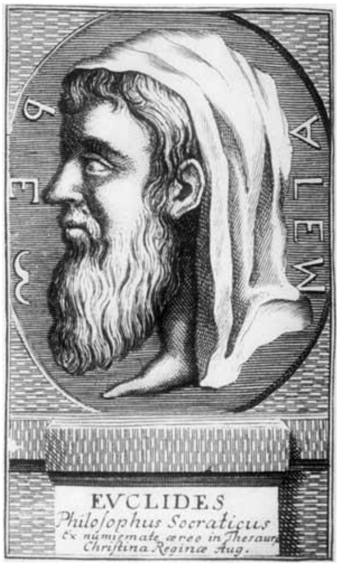
A study of geometrical and physical optics. Prerequisite: PHYS 222. Even years Fall.

Therefore, we concentrate on 1 and 2. The three levels can be compared to three levels of growing up: childhood, adolescence, and maturity. Each level that follows a previous level incorporates the previous level. In this sense, geometrical optics is the most limited area and only valid in certain circumstances. Such circumstances include much – mirrors, lenses, the camera, eyeglasses, kaleidoscopes, etc. And it would be a waste of time and energy to try to understand these applications with the wave or quantum machinery. The next level, physical optics, helps us understand diffraction and interference. The most advanced phase, quantum optics, goes beyond and we can understand the photoelectric effect and begin to enter the realm of quantum field theory for electrodynamics.

After teaching *Humanities 324 The Modern World* at UNCA for 20 semesters, I like to compare Wordsworth's three levels of appreciation of nature in *Tintern Abbey* with the three levels of optics. Wordsworth says the first level is like a doe bounding through the woods. This level is like playing hide-and-seek in the forest. The second level that comes when one gets older is appreciating nature for its sublime beauty. Now we have an emotional feeling towards the forest as we look at the trees and paths. Finally, Wordsworth indicates that the third and highest level is where Nature becomes our moral guardian and teacher. Here is where if you have a question about life, you walk in the woods for an hour or two to receive your answer. Often, I use this method to help me solve problems in physics and come up with ideas. Once walking from my car at UNCA near the wooded areas on my way to Robinson Hall, I came up with an idea for a publication

Before we go on, let me make one final comment of how mature physics frameworks embrace the old. Einstein's relativity reduces to Newton when speeds are small compared to the speed of light. So in this sense Einstein's relativity includes Newton. Similarly the adult with her deep connection to Nature (level 3) can take time to enjoy the beauty of the forest (level 2) and play with her younger sister hide and seek, or roll in the grass with a puppy (level 1).

A1. Light Travels in Straight Lines



Euclid of Alexandria (c. 325 BCE – c. 265 BCE)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

The idea that light travels in straight lines dates back to the Greek Euclid of Alexandria (Egypt), also known simply as Euclid. Euclid is also known as the founder of geometry which we know today as Euclidean geometry. Such geometry is your standard flat geometry you learn in high school.

It is therefore proper that we refer to light traveling in straight lines as geometrical optics, a geometry of light. See Fig. 2 below for the classic demonstration that light travels in straight lines using lasers and chalk dust so we can see the beams.

Fig. 2. Light Travels in Straight Lines. Scattering off chalk dust, Spring 2020.
Video: https://youtu.be/Hr5MxFnioow

I would like to show you the beautiful mathematics of the calculus of variations to demonstrate that the shortest distance between two points is a straight line. This exercise is very powerful since the calculus of variations is extremely important in theoretical physics and mathematics. Doing a simple problem with this tool will help us master the method.

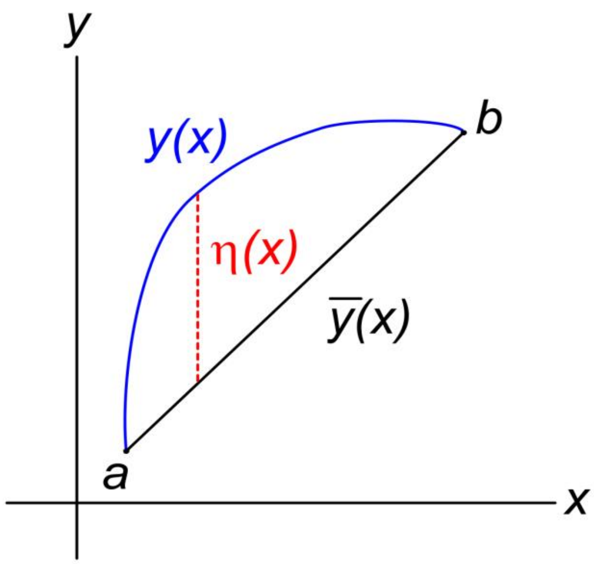


Fig. 3. Two Paths from a to b.

See Fig. 3 for two paths from a to b. Let the shortest path be $\bar{y}(x)$ and some general and arbitrary longer path be $y(x)$. The deviation of $y(x)$ from the ideal path is $\eta(x)$. Therefore,

$$y(x) = \bar{y}(x) + \eta(x)\,.$$

The arc length for a small segment of a general curve is show in Fig. 4.

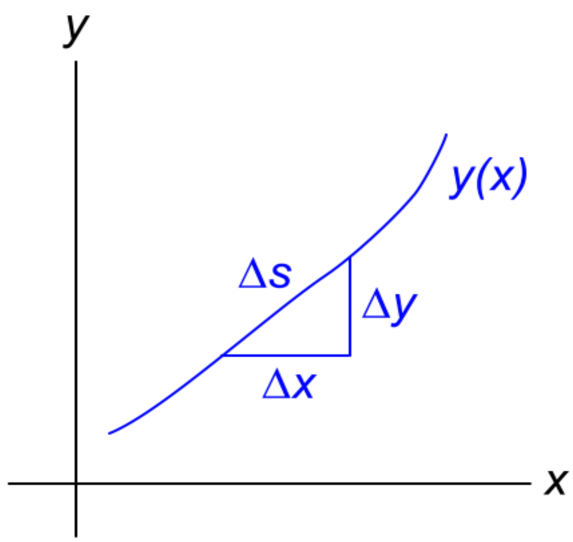


Fig. 4. General Curve with Arc length.

The arc length segment Δs can be written as

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

using the Pythagorean Theorem. We can also modify this expression to obtain:

$$\Delta s = \sqrt{1 + (\frac{\Delta y}{\Delta x})^2}\ \Delta x\,.$$

In the limit of differentials we have $ds = \sqrt{1 + (\frac{dy}{dx})^2}\ dx = \sqrt{1 + (y')^2}\ dx$, where $y' = \frac{dy}{dx}$, the derivative. The total arc length from x_a to x_b, or simply a to b, is given by the integral

$$S = \int_a^b ds = \int_a^b \sqrt{1 + (y')^2}\ dx\,.$$

Now we substitute $y(x) = \bar{y}(x) + \eta(x)$ into $S = \int_a^b ds = \int_a^b \sqrt{1+(y')^2}\, dx$ and arrive at

$$S = \int_a^b \sqrt{1+(\bar{y}'+\eta')^2}\, dx$$

for the general case. Since the $\eta(x)$ will be small when we are close to the ideal path, we will neglect the quadratic term compared to the linear term as we work out the squared term under the square-root sign.

$$(\bar{y}'+\eta')^2 = (\bar{y}')^2 + 2\bar{y}'\eta' + (\eta')^2 \approx (\bar{y}')^2 + 2\bar{y}'\eta'$$

Then,

$$S = \int_a^b \sqrt{1+(\bar{y}')^2 + 2\bar{y}'\eta'}\, dx .$$

The next step is to consider

$$\sqrt{1+(\bar{y}')^2 + 2\bar{y}'\eta'} = \sqrt{1+(\bar{y}')^2}\sqrt{1+\frac{2\bar{y}'\eta'}{1+(\bar{y}')^2}} ,$$

so that we can have the ideal path square-root component factored out and we can now expand the square-root component at the far right. We take

$\sqrt{1+\frac{2\bar{y}'\eta'}{1+(\bar{y}')^2}}$ and use $(1+\varepsilon)^n \approx 1 + n\varepsilon$ where $\varepsilon = \frac{2\bar{y}'\eta'}{1+(\bar{y}')^2}$ and $n = \frac{1}{2}$ to obtain $\sqrt{1+\frac{2\bar{y}'\eta'}{1+(\bar{y}')^2}} = 1 + \frac{1}{2}\frac{2\bar{y}'\eta'}{1+(\bar{y}')^2} = 1 + \frac{\bar{y}'\eta'}{1+(\bar{y}')^2}$.

The entire integrand $\sqrt{1+(\bar{y}')^2 + 2\bar{y}'\eta'}$ then becomes

$$\sqrt{1+(\bar{y}')^2+2\bar{y}'\eta'} = \sqrt{1+(\bar{y}')^2}\sqrt{1+\frac{2\bar{y}'\eta'}{1+(\bar{y}')^2}} = \sqrt{1+(\bar{y}')^2}\left[1+\frac{\bar{y}'\eta'}{1+(\bar{y}')^2}\right]$$

and the integral is

$$S = \int_a^b \sqrt{1+(\bar{y}')^2}\left[1+\frac{\bar{y}'\eta'}{1+(\bar{y}')^2}\right] dx \, .$$

This integral has two components. The arc length for the perfect ideal path and some error piece.

$$S = \int_a^b \sqrt{1+(\bar{y}')^2}\left[1+\frac{\bar{y}'\eta'}{1+(\bar{y}')^2}\right] dx = \bar{S} + \delta S$$, where

$$\bar{S} = \int_a^b \sqrt{1+(\bar{y}')^2}\, dx$$ and $$\delta S = \int_a^b \frac{\bar{y}'\eta'}{\sqrt{1+(\bar{y}')^2}}\, dx \, .$$

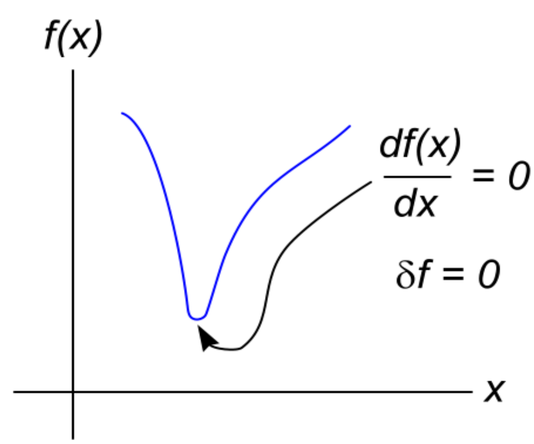


Fig. 5. Max-Min Problem in Calculus.

When the ideal path $\bar{S}$ is realized, $\delta S = 0$. We have then minimized the path. Recall max-min problems from calculus. See the figure at the right. You take a derivative and set it equal to zero to find an extremum. In this case it is a minimum. Note that the variation in the function, i.e., $\delta f = 0$, at an extremum is zero. In a similar way, we achieve the extremum when $\delta S = 0$.

Therefore,

$$\delta S = \int_a^b \frac{\bar{y}'\eta'}{\sqrt{1+(\bar{y}')^2}}\, dx = 0$$, which we can write as

$$\delta S = \int_a^b \frac{\bar{y}'}{\sqrt{1+(\bar{y}')^2}} \frac{d\eta}{dx}\, dx = 0 \, .$$

The last step is to note that $\eta = \eta(x)$ is the deviation from the ideal path for an arbitrary wrong path, i.e., not the ideal path, but with the condition that you must start at the point "a" and end up at "b", i.e.,

$$\eta_a = \eta(x_a) = 0 \quad \text{and} \quad \eta_b = \eta(x_b) = 0 \, .$$

So we need to lift that derivative off the $\dfrac{d\eta}{dx}$ to isolate the arbitrary errors η. All the books will tell you integration by parts. Well, yes, they are right. But I say the product rule:

$$\frac{d}{dx}\left[\frac{\bar{y}'}{\sqrt{1+(\bar{y}')^2}}\eta\right] = \frac{d}{dx}\left[\frac{\bar{y}'}{\sqrt{1+(\bar{y}')^2}}\right]\eta + \frac{\bar{y}'}{\sqrt{1+(\bar{y}')^2}}\frac{d\eta}{dx} \, .$$

Then, our integrand $\dfrac{\bar{y}'}{\sqrt{1+(\bar{y}')^2}}\dfrac{d\eta}{dx}$, the last term in the above equation, can be written as

$$\frac{d}{dx}\left[\frac{\bar{y}'}{\sqrt{1+(\bar{y}')^2}}\eta\right] - \frac{d}{dx}\left[\frac{\bar{y}'}{\sqrt{1+(\bar{y}')^2}}\right]\eta$$

, and we have two integrals:

$$\delta S = \int_a^b \frac{d}{dx}\left[\frac{\bar{y}'}{\sqrt{1+(\bar{y}')^2}}\eta\right] dx - \int_a^b \frac{d}{dx}\left[\frac{\bar{y}'}{\sqrt{1+(\bar{y}')^2}}\right]\eta\, dx = 0 \, .$$

The first integral is trivial, as is the case with integration by parts, which is what we are doing. It is just easier to think product rule for differentiation – easier to remember rather than that let u equal this function and dv equal that other part. Furthermore, the first integral

$$\int_a^b \frac{d}{dx}\left[\frac{\bar{y}'}{\sqrt{1+(\bar{y}')^2}}\eta\right]dx = \frac{\bar{y}'}{\sqrt{1+(\bar{y}')^2}}\eta\Bigg|_a^b = 0$$

since $\eta_a = \eta(x_a) = 0$ and $\eta_b = \eta(x_b) = 0$ at the endpoints. This result leaves us with

$$\delta S = -\int_a^b \frac{d}{dx}\left[\frac{\bar{y}'}{\sqrt{1+(\bar{y}')^2}}\right]\eta\, dx = 0 .$$

Since the η function is an arbitrary error function, to make the integral always zero, the stuff multiplying it must vanish:

$$\frac{d}{dx}\left[\frac{\bar{y}'}{\sqrt{1+(\bar{y}')^2}}\right] = 0 .$$

The conclusion is that $\frac{\bar{y}'}{\sqrt{1+(\bar{y}')^2}} = const \equiv K$, i.e., a constant. The only way to obtain a constant is that the derivative for the ideal path must be also be a constant:

$\bar{y}' = \frac{d\bar{y}}{dx} = m$, where m is a constant.

You know this from inspection of $\frac{\bar{y}'}{\sqrt{1+(\bar{y}')^2}} = const \equiv K$. But if you want to take the long way, you can express $\bar{y}'$ in terms of the constant K. First square things, $\frac{(\bar{y}')^2}{1+(\bar{y}')^2} = K^2$. Then arrange to find $(\bar{y}')^2 = K^2[1+(\bar{y}')^2]$ and $(\bar{y}')^2\left[1-K^2\right] = K^2$. Finally $(\bar{y}')^2 = \frac{K^2}{1-K^2}$ and $y' = \sqrt{\frac{K^2}{1-K^2}} \equiv m$.

Therefore, the ideal path is a straight line

$$\bar{y} = mx + b,$$

where m is the slope and b is the y-intercept. At this stage we can set $y = \bar{y}$ as we have found the correct path, the path where the errors η along the path are zero. The final result is

$$y = mx + b.$$

Now granted this analysis has a lot of steps just to find out something we already knew. But along the way, you learned a lot of nice mathematics. Or perhaps for some of you, it was a review from something similar you encountered in a calculus class or if you took my Theoretical Physics course. The subject is called the calculus of variations. You may run into variations on this theme.

The approach I took was based on Feynman's approach to the calculus of variations he used in an introductory course at Caltech in the early 1960s to minimize the action in physics. Historically, many mathematicians contributed to the calculus of variations. A famous form in physics is the use of the calculus of variations to minimum the action and arrive at the Euler-Lagrange equations, a fancy version of F = ma.

We minimized $S = \int_a^b ds = \int_a^b \sqrt{1+(y')^2}\, dx$, which has the form $S = \int_a^b L(y')\, dx$. The more general case $S = \int_a^b L(y, y')\, dx$ leads to the Euler-Lagrange formula

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial x}.$$

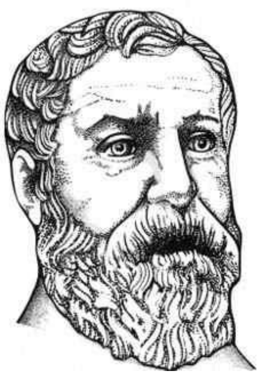
For our case there is no explicit dependence of our $L = \sqrt{1+(y')^2}$ on x, so the Euler-Lagrange equations reduces to

$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial x}$, which leads to our previous result $\frac{d}{dx}\left[\frac{\bar{y}'}{\sqrt{1+(\bar{y}')^2}}\right] = 0$. When you use the Lagrangian $L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - V(x)$, one obtains

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=\frac{\partial L}{\partial x}$$, which gives $$\frac{d}{dt}(m\dot{x})=-\frac{dV(x)}{dx}$$, i.e., $ma=F$.

At some point in your undergraduate physics studies you should see this worked out. I have reproduced the Feynman derivation in my *Theoretical Physics* course *Chapter W. The Principle of Least Action.* You can find the lecture here: https://youtu.be/0Jwcyso6saI

A2. The Principle of Least Time



Heron of Alexandria (c. 10 – c. 75)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

Heron of Alexandria, also Hero of Alexandria, a Greek mathematician and inventor, born in Alexandria, Egypt, promoted that idea that light takes the shortest path between two points. We can view this principle as light taking the least time.

But we had to wait until Fermat to have a method to explore light traveling in different media.

Pierre de Fermat (1601-1665), France
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

The Principle of Least Time is often referred to Fermat's Principle of Least time. Fermat, by the way was the famous French mathematician with his last theorem: Fermat's Last Theorem. The theorem states that

$$a^n+b^n=c^n ,$$

where a , b , and c are positive integers, can work for $n=1$ and $n=2$, but for no other integers $n\geq 3$. The case $n=1$ is the trivial case where solutions like $3^1+4^1=7^1$ work and the case $n=2$ is the famous Pythagorean theorem. An example of the latter is $3^2+4^2=5^2$. But no solutions for everything else? What? We found out about his claim in a note he wrote in a book: "I have discovered a truly remarkable proof of this theorem which this margin is too small to contain." This theorem baffled mathematicians for three and a half centuries. Then, in the 1990s it was proven, using methods that were discovered after Fermat's time. Did Fermat sensed the truth of his theorem by intuition? Or did he have a flawed argument leading to the claim? In any case, Fermat was right. The remarkable claim has been proven.

We now proceed to a least time problem, one involving two media. But we will complete the analysis in the next class. See Fig. 6 for a lifeguard about to reach a drowning person in the water. Which path would you sketch in order for the “Lifeguard Dinosaur” to reach the “Save Me!” drowning person in the least time? Note that the dinosaur Dino runs faster on the sand compared to swimming in the water.

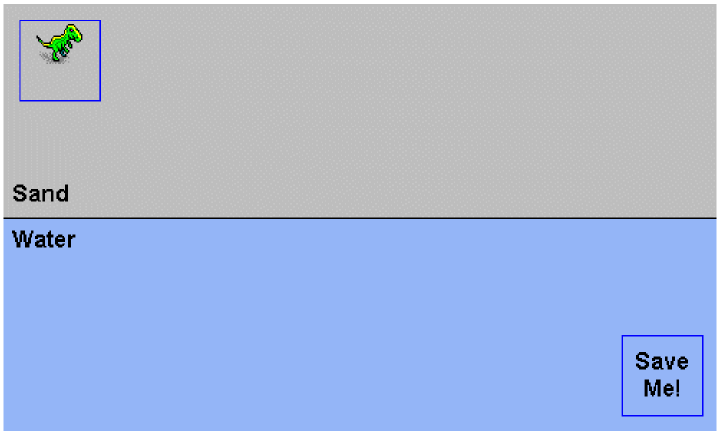


Fig. 6. A Lifeguard Getting to the Drowning Person in the Shortest Time.

We will complete the analysis of this problem in the next chapter. But note how the ants solve the problem below in Fig. 7.

Fig. 7. Ants traveling faster in the white Region, then slower in the green region as they travel to reach the food in the lower right corner.
Courtesy Simon Tragust